Approximation by an iterative method of a low Mach model with temperature dependent viscosity

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Abstract

In this work, we prove the existence and the uniqueness of the strong solution of a low-Mach model, for which the dynamic viscosity of the fluid is a given function of its temperature. The method is based on the convergence study of a sequence towards the solution, for which the rates are also given. The originality of the approach is to consider the system in terms of the temperature and the velocity, leading to a non-linear temperature equation and the development of some specific tools and results.

Introduction

During the last three decades, many studies have been devoted to the Boussinesq model arising from a zeroth-order approximation of the coupling between the Navier-Stokes and the thermodynamic equations. Nevertheless, in many combustion processes as well as in convective–conductive heat transferts, the sound waves speed can be much faster than the entropy or the vorticity ones. For such phenomena, the Boussinesq incompressible approximation can become irrelevant when the compressibility effects can no more be neglected because of large variations of temperature and density, even if pressure ones are much smaller.

Alternatively, the low Mach model allows to generate intermediate solutions between the compressible Navier-Stokes model and the incompressible Navier-Stokes one. In [29], Majda and Sethian introduce a limiting system which describes combustion processes at low Mach number in a confined region and solve it numerically. These equations allow to treat large temperature and density variations, associated to substantial interactions with the hydrodynamic flow field. However, these models are considerably simpler than the complete system of compressible combustion equations because of the cancelation of the acoustic

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waves effects, removed from the set of equations.

The theoretical study of the low Mach number limit is a vast subject. The involved equations may be isentropic or non-isentropic, the fluid may be viscous or inviscid, the temperature variations may be small or large, the domain may be bounded or unbounded. Considering the compressible isentropic Navier-Stokes equations, the reader may refer to the major breakthrough of Lions [26] where the global existence of weak solutions with finite energy is considered. In [16], the authors present a survey of the mathematical properties of solutions to the Navier-Stokes equations for compressible fluids: existence and regularity, uniqueness without assumption on the initial data, as well as a counterexample which gives some evidence for finite time blow-up for local smooth initial solutions. Many others results are devoted to the existence of solutions for inviscid fluids: see for instance [2, 22, 30, 31] and references therein.

The justification of the incompressible Navier-Stokes equations as the zero-Mach limit of the compressible Navier-Stokes equation has been considered in [10–12, 14, 15, 27]. In [3–5], the author performs a rigorous analysis for the full Navier-Stokes system with large temperature variations in the Sobolev spaces H^s, with s large enough. In [7, 23], the authors investigate the problem of global existence in time of weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids, where the viscosity and conductivity coefficients depend on the density and the temperature.

To our knowledge, [17] and [13] are the only works dedicated to the theoretical study of the low Mach number limit system (1)-(2)-(3)-(4). In [17], Emdid establishes the local-in-time existence of smooth solutions in Sobolev spaces. In [13], Danchin and Liao study the well-posedness issue in the critical Besov spaces, locally and globally, assuming that the initial density is close to a constant and that the initial velocity is small enough. It was observed by Lions in [26] that a small perturbation of a constant initial density provides a global existence of weak solutions in the case of two-dimensional domains. In a more recent work [9], Bresch and co-authors introduce a new mathematical entropy and show the global existence of weak solutions in the three-dimensional case with no smallness assumption on the initial velocity.

In this work, we are interested in a specific low-Mach model proposed in [8], for which the dynamic viscosity of the fluid is a specific function of the density. Here, the model is reformulated in terms of the temperature and the velocity. Our goal is to prove the existence and the uniqueness of the strong solution of the problem as a limit of a given sequence, and to obtain some results on the convergence rates. Let us note that the paper follows a similar progression as the one proposed in [20, 21]. The originality of the present contribution is mainly based on the fact that we choose to consider the temperature unknown instead of the density one in the set of equations. Consequently, the temperature equation becomes non-linear, and some specific results need to be established. Throughout the paper, we will mainly focus on these particularities.

The sketch of the paper is the following. Section 1 starts by deriving the low-Mach model under consideration in term of the velocity $\mathbf{v}$, the pressure p and the temperature ϑ. Then, some space functions and norms are defined. The iterative scheme is given, as well as the main results of the paper (Theorems 1 and 2). Section 2 is devoted to some analytical tools needed to prove these results. Section 3 gives some existence and uniqueness results as well as *a priori* estimates on the sequence $(\mathbf{v}^n, p^n, \vartheta^n)$ induced by the iterative scheme. Finally, section 4 establishes the convergence of the sequence towards the strong solution of the initial system, corresponding to the proofs of Theorems 1 and 2.

1 Preliminaries

1.1 Model derivation

Formally, a low Mach number model is obtained considering an asymptotic development of the pressure with respect to the Mach number M, representing the ratio of the flow velocity to the local speed of sound. Consequently, the conservation of momentum equation indicates that the pressure can be considered constant in space up to a term of order M^2. We assume moreover that the density and the temperature of the fluid are linked by the ideal gaz law, so that the one can be deduced from the other.

Let $\Omega \subset \mathbb{R}^3$ be a bounded regular open domain with a boundary Γ. We denote by $\boldsymbol{n}$ the unit outward normal on the boundary Γ and by $[0,T]$ a given time interval, where $T > 0$. We also introduce the notations $\mathcal{Q}_T = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \Gamma$. The fluid is described by its density $\rho(t,\mathbf{x}) \in \mathbb{R}_+^*$ at point $(t,\mathbf{x}) \in \mathcal{Q}_T$, its mean mass velocity $\mathbf{u}(t,\mathbf{x}) \in \mathbb{R}^3$, its temperature $\vartheta(t,\mathbf{x}) \in \mathbb{R}_+^*$ as well as its pressure $\pi(t,\mathbf{x}) \in \mathbb{R}$. Considering the full Navier-Stokes system and assuming that

$$\pi(t,\mathbf{x}) = P(t) + M^2 q(t,\mathbf{x}) + o(M^2),$$

where the thermodynamic pressure $P(t) = P_0 > 0$ is constant for all $t \geq 0$ (and not only independent of $\mathbf{x}$), we can deduce the following model in $\mathcal{Q}_T$ by letting M go to 0 (see [13, 25]):

$$\partial_t \rho + \mathrm{div}\,(\rho \mathbf{u}) = 0, \tag{1}$$

$$\partial_t(\rho\mathbf{u}) + \mathrm{div}\,(\rho\mathbf{u} \otimes \mathbf{u}) - \mathrm{div}\,\big(2\tilde{\mu}\,\mathbb{D}(\mathbf{u}) + \tilde{\lambda}\,\mathrm{div}\,\mathbf{u}\,\mathbb{I}\big) + \nabla q = \rho\mathbf{f}, \tag{2}$$

$$\gamma P_0 \mathrm{div}\,\mathbf{u} = (\gamma - 1)\mathrm{div}\,(k\nabla\vartheta), \tag{3}$$

$$P_0 = R\rho\vartheta. \tag{4}$$

Here, $\mathbb{D}(\mathbf{u}) = (\nabla\mathbf{u} + \nabla^t\mathbf{u})/2$ denotes the deformation tensor, $\mathbb{I}$ the identity matrix and $\mathbf{f}(t,\mathbf{x}) \in \mathbb{R}^3$ the external forces. Given a vector $\boldsymbol{a} \in \mathbb{R}^3$ and a matrix valued function A, the values of $\mathrm{div}\,\boldsymbol{a}$ and $\mathrm{div}\,A$ are respectively the scalar and the vector quantities defined by

$$\mathrm{div}\,\boldsymbol{a} = \sum_{j=1}^{3} \partial_{x_j}\boldsymbol{a}_j \quad \text{ and } \quad (\mathrm{div}\,A)_i = \sum_{j=1}^{3} \partial_{x_j} A_{ij}, \quad 1 \leq i \leq 3.$$

The heat conductivity k and the viscosity coefficients $\tilde{\mu}$ and $\tilde{\lambda}$ may depend smoothly on ρ and ϑ, and fulfill

$$k > 0, \quad \tilde{\mu} > 0, \quad 3\tilde{\lambda} + 2\tilde{\mu} \geq 0.$$

Since we deal with ideal gas (see (4)), γ is defined by

$$\gamma = 1 + \frac{R}{C_v} \in (1, 5/3],$$

where R and C_v stand respectively for the ideal gas constant and the specific heat constant. We underline that the non-standard constraint (3) is easily deduced from the conservation of the energy.

We remark that the system (1)-(2)-(3) is of parabolic type, where the pressure term can be seen as a Langrange multiplier corresponding to the non-standard constraint (3). In order to reduce the study to a system more similar to the incompressible Navier-Stokes problem for the temperature and the momentum equations, a solenoidal velocity $\mathbf{v}(t,\mathbf{x}) \in \mathbb{R}^3$ is introduced and defined by

$$\mathbf{v} = \mathbf{u} - \lambda \nabla \vartheta, \tag{5}$$

where $\lambda = \dfrac{(\gamma-1)k}{\gamma P_0} > 0$ is a fixed constant. Then (3) becomes

$$\operatorname{div} \mathbf{v} = 0.$$

Following the idea introduced in [8] where a particular relation between the density and the viscosity in the combustion model is introduced, we set $3\tilde{\lambda} + 2\tilde{\mu} = 0$ and we assume morerover that

$$\tilde{\mu}(\vartheta) = \frac{P_0}{R}\mu(\vartheta)$$

with

$$\mu(\vartheta) = -\lambda \ln \vartheta, \tag{6}$$

so that the viscosity $\mu(\vartheta)$ is strictly positive if and only if $\vartheta \in (0,1)$. We can now introduce the model we are going to work on.

Lemma 1. *Using relations* (5) *and* (6)*, the system* (1)-(2)-(3)-(4) *can be rewritten as the following one:*

$$\begin{aligned} \partial_t \vartheta + \mathbf{v}\cdot\nabla\vartheta + 2\lambda|\nabla\vartheta|^2 - \lambda \operatorname{div}(\vartheta\nabla\vartheta) &= 0, &(7)\\ \frac{1}{\vartheta}(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) - \operatorname{div}(\mu(\vartheta)\nabla\mathbf{v}) + \frac{\lambda}{\vartheta}(\nabla\mathbf{v} - \nabla^t\mathbf{v})\nabla\vartheta + \nabla p &= \frac{1}{\vartheta}\mathbf{f}, &(8)\\ \operatorname{div}\mathbf{v} &= 0, &(9)\\ \rho &= \frac{P_0}{R\vartheta}, \end{aligned}$$

with the modified pressure p defined by

$$p = \frac{R}{P_0}q + \lambda^2\Delta\vartheta - \frac{2\lambda}{3}\mu(\vartheta)\Delta\vartheta + \lambda^2 \ln\vartheta\Delta\vartheta.$$

Proof. Using the state equation (4) in the mass conservation (1), we obtain

$$\partial_t\vartheta + \mathbf{u}\cdot\nabla\vartheta - \vartheta \operatorname{div}\mathbf{u} = 0,$$

and from the definition of the solenoidal velocity $\mathbf{v}$ in (5) we get

$$\partial_t\vartheta + \mathbf{v}\cdot\nabla\vartheta + \lambda|\nabla\vartheta|^2 - \lambda\vartheta \operatorname{div}\mathbf{v} = 0,$$

or equivalently (7). Using now (1) and (4) in the momentum equation (2), we write

$$\frac{1}{\vartheta}(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) + \frac{R}{P_0}\nabla q - \frac{R}{P_0}\operatorname{div}(2\tilde{\mu}(\vartheta)\mathbb{D}(\mathbf{u})) - \frac{R}{3P_0}\nabla(2\tilde{\mu}(\vartheta)\operatorname{div}\mathbf{u}) = \frac{\mathbf{f}}{\vartheta}. \tag{10}$$

Developping each term of (10) by using (7) yields

$$\begin{aligned}\frac{1}{\vartheta}\partial_t\mathbf{u} &= \frac{1}{\vartheta}\partial_t\mathbf{v} + \frac{\lambda}{\vartheta}\partial_t(\nabla\vartheta)\\ &= \frac{1}{\vartheta}\partial_t\mathbf{v} + \frac{\lambda}{\vartheta}\nabla\left(\lambda\vartheta\Delta\vartheta - \mathbf{v}\cdot\nabla\vartheta - \lambda|\nabla\vartheta|^2\right)\\ &= \frac{1}{\vartheta}\partial_t\mathbf{v} + \frac{\lambda^2}{\vartheta}\Delta\vartheta\nabla\vartheta + \lambda^2\nabla\Delta\vartheta - \frac{\lambda}{\vartheta}\nabla(\mathbf{v}\cdot\nabla\vartheta) - 2\frac{\lambda^2}{\vartheta}(\nabla\vartheta\cdot\nabla)\nabla\vartheta,\end{aligned}$$

where

$$\nabla(\mathbf{v}\cdot\nabla\vartheta) = (\nabla\vartheta\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\nabla\vartheta + (\nabla^t\mathbf{v} - \nabla\mathbf{v})\nabla\vartheta.$$

Moreover,

$$\frac{1}{\vartheta}(\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{\vartheta}(\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\lambda}{\vartheta}(\nabla\vartheta\cdot\nabla)\mathbf{v} + \frac{\lambda}{\vartheta}(\mathbf{v}\cdot\nabla)\nabla\vartheta + \frac{\lambda^2}{\vartheta}(\nabla\vartheta\cdot\nabla)\nabla\vartheta.$$

For the viscosity terms, we have

$$\begin{aligned}-\frac{R}{P_0}\mathrm{div}\,(2\tilde{\mu}(\vartheta)\mathbb{D}(\mathbf{u})) &= -\mathrm{div}\,(\mu(\vartheta)\nabla\mathbf{v}) - \lambda\mathrm{div}\,(\mu(\vartheta)\nabla\nabla\vartheta)\\ &= -\mathrm{div}\,(\mu(\vartheta)\nabla\mathbf{v}) - \lambda\mu'(\vartheta)(\nabla\vartheta\cdot\nabla)\nabla\vartheta - \lambda\mu(\vartheta)\Delta\nabla\vartheta,\end{aligned}$$

and

$$-\frac{R}{3P_0}\nabla(2\tilde{\mu}(\vartheta)\mathrm{div}\,\mathbf{u}) = -\frac{2\lambda}{3}\nabla(\mu(\vartheta)\Delta\vartheta).$$

Then (10) can be rewritten as

$$\begin{aligned}&\frac{1}{\vartheta}\partial_t\mathbf{v} + \frac{1}{\vartheta}(\mathbf{v}\cdot\nabla)\mathbf{v} - \mathrm{div}\,(\mu(\vartheta)\nabla\mathbf{v}) + \frac{\lambda}{\vartheta}(\nabla\mathbf{v} - \nabla^t\mathbf{v})\nabla\vartheta + \lambda^2\left(\frac{1}{\vartheta}\Delta\vartheta\nabla\vartheta - \frac{1}{\vartheta}(\nabla\vartheta\cdot\nabla)\nabla\vartheta\right)\\ &\quad -\lambda\mu'(\vartheta)(\nabla\vartheta\cdot\nabla)\nabla\vartheta - \lambda\mu(\vartheta)\Delta\nabla\vartheta + \nabla\left(\frac{R}{P_0}q + \lambda^2\Delta\vartheta - \frac{2\lambda}{3}\mu(\vartheta)\Delta\vartheta\right) = \frac{1}{\vartheta}\mathbf{f}.\end{aligned} \tag{11}$$

Finally, by using (6) we obtain

$$\begin{aligned}-\lambda\mu'(\vartheta)(\nabla\vartheta\cdot\nabla)\nabla\vartheta - \lambda\mu(\vartheta)\Delta\nabla\vartheta &= \frac{\lambda^2}{\vartheta}(\nabla\vartheta\cdot\nabla)\nabla\vartheta + \lambda^2\ln\vartheta\Delta\nabla\vartheta\\ &= \frac{\lambda^2}{\vartheta}(\nabla\vartheta\cdot\nabla)\nabla\vartheta + \lambda^2\nabla\left(\ln\vartheta\Delta\vartheta\right) - \frac{\lambda^2}{\vartheta}\Delta\vartheta\nabla\vartheta.\end{aligned}$$

Introducing this expression in in (11) and including all the gradient terms in the definition of ∇p, we obtain (8). □

The combustion model (7)–(8)–(9) is completed with the following boundary and initial conditions (see [17]):

$$\frac{\partial\vartheta}{\partial\boldsymbol{n}}(t,\mathbf{x}) = 0, \qquad \mathbf{v}(t,\mathbf{x}) = 0, \qquad \forall\,(t,\mathbf{x})\in\Sigma, \tag{12}$$

$$\vartheta(0,\mathbf{x}) = \vartheta_0(\mathbf{x}), \qquad \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}), \quad \forall\,\mathbf{x}\in\Omega, \tag{13}$$

where $\vartheta_0 : \Omega \to \mathbb{R}$ and $\mathbf{v}_0 : \Omega \to \mathbb{R}^3$ are given functions, with $\mathrm{div}\,\mathbf{v}_0 = 0$.

1.2 Space functions and equivalent norms

We now introduce the standard functional spaces for the Navier-Stokes framework:

$$
\begin{aligned}
\mathbb{L}^2(\Omega) &= L^2(\Omega)^3,\\
\mathbb{H}^1_0(\Omega) &= H^1_0(\Omega)^3,\\
\mathcal{V} &= \left\{\mathbf{v}\in\mathcal{D}(\Omega)^2:\ \operatorname{div}\mathbf{v}=0 \text{ in } \Omega\right\},\\
\mathbf{V} &= \left\{\mathbf{v}\in\mathbb{H}^1_0(\Omega):\ \operatorname{div}\mathbf{v}=0 \text{ in } \Omega\right\},\\
\mathbf{H} &= \left\{\mathbf{v}\in\mathbb{L}^2(\Omega):\ \operatorname{div}\mathbf{v}=0 \text{ in } \Omega,\ \mathbf{v}\cdot\mathbf{n}=0 \text{ on } \Gamma\right\},\\
L^2_0(\Omega) &= \left\{p\in L^2(\Omega):\ \int_\Omega p(\boldsymbol{x})\ d\mathbf{x}=0\right\}.
\end{aligned}
$$

We recall that $\mathbf{V}$ and $\mathbf{H}$ are the closures of $\mathcal{V}$ in $\mathbb{H}^1_0(\Omega)$ and $\mathbb{L}^2(\Omega)$ respectively, and $(.,.)$ will denote the scalar product in $\mathbb{L}^2(\Omega)$ or $L^2(\Omega)$. Let $\mathbb{P}$ be the orthogonal projection of $\mathbb{L}^2(\Omega)$ onto $\mathbf{H}$ and $\mathbb{A}=-\mathbb{P}\Delta$ the Stokes operator defined in $D(\mathbb{A})=\mathbf{V}\cap\mathbb{H}^2(\Omega)$. We denote by $(\mathbf{w}_k)_{k\in\mathbb{N}}$ and $(\alpha_k)_{k\in\mathbb{N}}$ respectively the eigenvectors and eigenvalues of the Stokes operator. It is well known that the sequence $(\mathbf{w}_k)_{k\in\mathbb{N}}$ is orthogonal with the inner products $(\cdot,\cdot)$ and $(\nabla\cdot,\nabla\cdot)$. The norms $\|\mathbf{v}\|_{H^1}$ and $\|\nabla\mathbf{v}\|_{L^2}$ are equivalent in $\mathbf{V}$, and $\|\mathbf{v}\|_{H^2}$ and $\|\mathbb{A}\mathbf{v}\|_{L^2}$ are equivalent in $D(\mathbb{A})$. Moreover, $\|p\|_{H^1}$ and $\|\nabla p\|_{L^2}$ are equivalent in $H^1(\Omega)\cap L^2_0(\Omega)$.

Then, we introduce the spaces specifically needed for the temperature ($k=2,3$):

$$
\begin{aligned}
H^k_N(\Omega) &= \left\{\vartheta\in H^k(\Omega):\ \frac{\partial\vartheta}{\partial\mathbf{n}}=0 \text{ on } \partial\Omega\right\},\\
H^k_{N,0}(\Omega) &= \left\{\vartheta\in H^k(\Omega):\ \frac{\partial\vartheta}{\partial\mathbf{n}}=0 \text{ on } \partial\Omega \text{ and } \int_\Omega\vartheta(\mathbf{x})=0\right\}.
\end{aligned}
$$

If $\vartheta(t,.)\in H^k_N(\Omega)$ for all $t\in[0,T]$, then we define

$$
\widetilde{\vartheta}(t,.)=\vartheta(t,.)-\overline{\vartheta}(t)\in H^k_{N,0}\quad\text{where}\quad\overline{\vartheta}(t)=\frac{1}{|\Omega|}\int_\Omega\vartheta(t,\mathbf{x})\ d\mathbf{x}. \tag{14}
$$

Thanks to the $H^2(\Omega)$ and $H^3(\Omega)$ regularity of the Poisson problem with Neumann boundary conditions (see [19]), the norms $\|\theta\|_{H^2}$ and $\|\Delta\theta\|_{L^2}$ are equivalent in $H^2_{N,0}(\Omega)$, and $\|\theta\|_{H^3}$ and $\|\nabla\Delta\theta\|_{L^2}$ are equivalent in $H^3_{N,0}(\Omega)$. Consequently, thanks to (14), the norms $\|\nabla\theta\|_{H^1}$ and $\|\Delta\theta\|_{L^2}$ are equivalent in $H^2_N(\Omega)$, and $\|\nabla\theta\|_{H^2}$ and $\|\nabla\Delta\theta\|_{L^2}$ are equivalent in $H^3_N(\Omega)$.

1.3 The iterative scheme

We follow the approach of F. Guillén-González and his co-authors for Kazhikhov-Smagulov type models with constant viscosity in [20] as well as with linear viscosity in [21]. It consists in building iteratively a sequence $(\mathbf{v}^n,p^n,\vartheta^n)$ of approximate solutions of (7)–(8)–(9) and (12)–(13) as follows:

- Initialization:
  - $\vartheta^0(t,\cdot)=\vartheta_0,\quad\forall t\in[0,T]$,
  - $\mathbf{v}^0(t,\cdot)=\mathbf{v}_0,\quad\forall t\in[0,T]$.

- Then, given $\mathbf{v}^{n-1}$ and ϑ^{n-1} :

  - ϑ^n is solution of

$$\begin{cases} \partial_t \vartheta^n + \mathbf{v}^{n-1} \cdot \nabla \vartheta^n + 2\,\lambda \nabla \vartheta^{n-1} \cdot \nabla \vartheta^n - \lambda \operatorname{div}(\vartheta^{n-1} \nabla \vartheta^n) = 0, \\ \vartheta^n(0,\cdot) = \vartheta_0, \quad \nabla \vartheta^n \cdot \mathbf{n}|_{\partial\Omega} = 0. \end{cases} \tag{15}$$

  - $\mathbf{v}^n$ and p^n are solutions of

$$\begin{cases} \dfrac{1}{\vartheta^n} \partial_t \mathbf{v}^n + \dfrac{1}{\vartheta^n} (\mathbf{v}^{n-1} \cdot \nabla) \mathbf{v}^n - \operatorname{div}(\mu(\vartheta^n) \nabla \mathbf{v}^n) \\ + \dfrac{\lambda}{\vartheta^n} (\nabla \mathbf{v}^n - \nabla^t \mathbf{v}^n) \nabla \vartheta^n + \nabla p^n = \dfrac{1}{\vartheta^n}\, \mathbf{f}, \\ \operatorname{div} \mathbf{v}^n = 0, \quad \mathbf{v}^n(0,\cdot) = \mathbf{v}_0, \quad \mathbf{v}^n|_{\partial\Omega} = 0. \end{cases} \tag{16}$$

1.4 Main results

The goal of the paper is to prove that the sequence $(\mathbf{v}^n, p^n, \vartheta^n)$ converges towards the solution $(\mathbf{v}, p, \vartheta)$ of problem (7)–(8)–(9) and (12)–(13) for some norms, and to give some estimates of the convergence rates. We assume that

$$\vartheta_0 \in H^2_N(\Omega), \quad \mathbf{v}_0 \in \mathbf{V}, \quad \mathbf{f} \in L^2(\mathcal{Q}_T), \tag{17}$$

and that there exist two real numbers m and M such that

$$0 < m \leq \vartheta_0 \leq M < 1 \text{ a.e. } x \in \Omega. \tag{18}$$

Furthermore, we suppose that there exist some positive constants K_1 and K_2 such that

$$\exp\left(\frac{C_1\, T}{\lambda^5\, m^3} \left(|\ln M|^{-2} K_1^2 + \lambda^4 K_2^2 + \lambda^6\, M^4 \right) \right) \lambda\, \|\Delta \vartheta_0\|^2_{L^2} \leq K_2, \tag{19}$$

$$\exp\left(C_2\, T\, \frac{M^4 \left(\lambda^2\, |\ln M|^4\, m^4 + K_2^2 \right)}{\lambda^7\, m^{11}\, \min(|\ln M|^7, |\ln M|^{11})} \left(|\ln M|^{-2} K_1^2 + \lambda^4\, K_2^2 \right) \right) \times \left(\left\| \sqrt{\mu(\vartheta_0)} \nabla \mathbf{v}_0 \right\|^2_{L^2} + \frac{M}{m^2}\, \|\mathbf{f}\|^2_{L^2(\mathcal{Q}_T)} \right) \leq K_1, \tag{20}$$

where C_1 and C_2 are constants only depending on the domain, respectively defined in Lemmas 9 and 10. Note that the smallness assumptions (19) and (20) are satisfied if the initial and source data $\|\Delta \vartheta_0\|_{L^2}$, $\|\nabla \mathbf{v}_0\|_{L^2}$ and $\|\mathbf{f}\|_{L^2(\mathcal{Q}_T)}$ or if the final time T are small enough.

We give now the two main Theorems established in the paper.

Theorem 1. *Let assume that the data satisfy assumptions* (17)-(18)-(19)-(20)*. Then, the system* (7)–(8)–(9) *associated to initial and boundary conditions* (12)–(13) *admits a unique strong solution* $(\mathbf{v}, p, \vartheta)$ *such that*

$$\begin{aligned} &\vartheta \in L^2(0,T; H^3_N(\Omega)) \cap L^\infty(0,T; H^2_N(\Omega)), \qquad \partial_t \vartheta \in L^2(0,T; H^1(\Omega)), \\ &\mathbf{v} \in L^2(0,T; D(\mathbb{A})) \cap L^\infty(0,T; \boldsymbol{V}), \qquad \partial_t \mathbf{v} \in L^2(0,T; \boldsymbol{H}), \\ &p \in L^2(0,T; L^2_0(\Omega)). \end{aligned}$$

Moreover, $(\mathbf{v}, p, \vartheta)$ is obtained as the limit of $(\mathbf{v}^n, p^n, \vartheta^n)$ when $n \to \infty$, and for all $t \in [0,T]$, one has:

$$\|(\vartheta^n - \vartheta)(t)\|_{H^1}^2 + \int_0^t \left(\|(\vartheta^n - \vartheta)(s)\|_{H^2}^2 + \|(\partial_t\vartheta^n - \partial_t\vartheta)(s)\|_{L^2}^2\right) \mathrm{d}s \le D\left[\frac{(B\,t)^n}{n!}\right]^{\frac{1}{2}} \tag{21}$$

$$\|(\mathbf{v}^n - \mathbf{v})(t)\|_{L^2}^2 + \int_0^t \|(\mathbf{v}^n - \mathbf{v})(s)\|_{H^1}^2 \;\mathrm{d}s \le D\left[\frac{(B\,t)^n}{n!}\right]^{\frac{1}{2}}, \tag{22}$$

where B and D are some constants independent of n (but depending on the data and on λ).

Without additional assumptions, we prove a second result.

Theorem 2. *Under the assumptions of Theorem 1, we have:*

$$\|(\vartheta^n - \vartheta)(t)\|_{H^2}^2 + \int_0^t \left(\|(\vartheta^n - \vartheta)(s)\|_{H^3}^2 + \|(\partial_t\vartheta^n - \partial_t\vartheta)(s)\|_{H^1}^2\right) \mathrm{d}s \;\le D\left[\frac{(B\,t)^{n-1}}{\sqrt{(n-1)!}}\right]^{\frac{1}{2}}, \tag{23}$$

$$\|(\mathbf{v}^n - \mathbf{v})(t)\|_{H^1}^2 + \int_0^t \left(\|(\mathbf{v}^n - \mathbf{v})(s)\|_{H^2}^2 + \|(\partial_t\mathbf{v}^n - \partial_t\mathbf{v})(s)\|_{L^2}^2\right) \mathrm{d}s \le D\left[\frac{(B\,t)^{n-1}}{\sqrt{(n-1)!}}\right]^{\frac{1}{2}}, \tag{24}$$

$$\int_0^t \|(p^n - p)(s)\|_{L^2}^2 \;\mathrm{d}s \le D\left[\frac{(B\,t)^{n-1}}{\sqrt{(n-1)!}}\right]^{\frac{1}{2}}, \tag{25}$$

where B and D are some constants independent of n (but depending on the data and on λ).

Note that the loss of convergence rates between Theorems 1 and 2 constitutes a difference compared with the results given in [20]. This is a consequence of the non-linearity of the temperature equation, as we will see below.

2 Useful tools

In this section, we state some results to be used in later sections. The first two Lemmas are estimates of Gronwall's type. Lemma 2 is classical, and Lemma 3 is immediately adapted from Lemma 2.2 in [20].

Lemma 2. *(Gronwall). Let a, b, c and d be positive and $L^1(0,T)$ functions satisfying*

$$a'(t) + b(t) \le c(t)\,a(t) + d(t) \quad a.e.\, t \in (0,T).$$

Then, for any $t \in (0,T)$, one has:

$$a(t) + \int_0^t b(s)\,\mathrm{d}s \le \left(a(0) + \int_0^t d(s)\mathrm{d}s\right) \exp\left(\int_0^t c(s)\,\mathrm{d}s\right).$$

Lemma 3. *(Gronwall with recurrence). Let (a_n), (b_n), (c_n), (d_n) and (e_n) be sequences of positive functions such that*

- *(a_n), (b_n) and (e_n) belongs to $L^1(0,T)$,*
- *(c_n) and (d_n) are respectively bounded in $L^1(0,T)$ and $L^2(0,T)$,*

- *There exists $A \in \mathbb{R}$ such that $a_n(0) \leq A$.*

We assume

$$a_n'(t) + b_n(t) \leq c_n(t)\, a_n(t) + d_n(t)\, a_{n-1}(t) + e_n(t) \quad a.e.\, t \in (0,T).$$

Then, there exist positive constants B and D independent on n (but depending on bounds of $\|c_n\|_{L^1(0,T)}$ and $\|d_n\|_{L^2(0,T)}$) such that for any $t \in (0,T)$ and for any $n \geq 1$, one has:

$$a_n(t) + \int_0^t b_n(s)\,\mathrm{d}s \leq D\left(A\,\mathrm{e}^{\frac{B\,t}{2}} + \left[\sum_{k=1}^{n} \|e_k\|^2_{L^1(0,t)} \frac{(B\,t)^{n-k}}{(n-k)!}\right]^{\frac{1}{2}} + \|a_0\|_{L^\infty(0,t)} \left[\frac{(B\,t)^n}{n!}\right]^{\frac{1}{2}} \right).$$

Remark 1. *In the following, we will have $\|e_k\|^2_{L^1(0,t)} \leq D\left[\frac{(B\,t)^{k-1}}{(k-1)!}\right]^{\frac{1}{2}}$. In this case, one has:*

$$a_n(t) + \int_0^t b_n(s)\,\mathrm{d}s \leq D\left(A\,\mathrm{e}^{\frac{B\,t}{2}} + \left[\frac{(B\,t)^{n-1}}{\sqrt{(n-1)!}}\right]^{\frac{1}{2}} + \|a_0\|_{L^\infty(0,t)} \left[\frac{(B\,t)^n}{n!}\right]^{\frac{1}{2}} \right).$$

The following result, due to J.-L. Lions [24], will be used to obtain the existence of the approximate temperature.

Lemma 4. *Let U and L be Hilbert spaces in the following configuration:*

$$U \subset L \equiv L' \subset U',$$

each space being dense in the following one and the injections being continuous. Let $a :]0,T[\times U \times U \to \mathbb{R}$ such that $a(t,\cdot,\cdot)$ is bilinear a.e. $t \in]0,T[$ and which satisfies:

- *$t \to a(t,u,v)$ is measurable $\forall\, u, v \in U$,*
- *$\exists\, M > 0$ such that $|a(t,u,v)| \leq M\,\|u\|_U\,\|v\|_U$ a.e. $t \in [0,T]$, $\forall u, v \in U$,*
- *$\exists\, \alpha > 0$ and $\gamma > 0$ such that $a(t,u,u) \geq \alpha\,\|u\|^2_U - \gamma\,\|u\|^2_L$ a.e. $t \in [0,T]$, $\forall\, u \in U$.*

Let $f \in L^2(0,T;U')$ and $u_0 \in U$. Finally, let us consider the following problem:

$$\begin{cases} < \partial_t u, v >_{U',U} + a(t,u,v) = < f, v >_{U',U} \quad a.e.\ t \in]0,T[, \forall v \in U, \\ u(0) = u_0. \end{cases} \tag{26}$$

Then, problem (26) admits a unique solution $u \in \mathcal{C}(0,T;L) \cap L^2(0,T,U)$ such that $\partial_t u \in L^2(0,T;U')$.

The following result, which is proved in [28], gives estimates for the pressure associated to the Helmholtz decomposition of the laplacian.

Lemma 5. *Let $\mathbf{v} \in D(\mathbb{A})$. We consider the Helmholtz decomposition of $-\Delta\mathbf{v}$, namely:*

$$-\Delta\mathbf{v} = \mathbb{A}\mathbf{v} + \nabla q, \tag{27}$$

where $q \in H^1(\Omega) \cap L^2_0(\Omega)$. Then, for all $\varepsilon > 0$, there exists a positive constant C only depending on the domain such that

$$\|q\|^2_{L^2} \leq \varepsilon\,\|\mathbb{A}\mathbf{v}\|^2_{L^2} + C\left(1 + \frac{1}{\varepsilon}\right)\|\nabla\mathbf{v}\|^2_{L^2}. \tag{28}$$

Finally, the following Lemma (see [18]) will be used to show the existence of the velocity.

Lemma 6. *(Continuity method). Let* $\mathbf{X}$ *be a Banach space, and* $\mathbf{Y}$ *a normed linear space. Let* L_0 *and* L_1 *be bounded linear operators from* $\mathbf{X}$ *into* $\mathbf{Y}$*. For each* $\alpha \in [0;1]$*, set* $L_\alpha = (1-\alpha)\, L_0 + \alpha\, L_1$*,and suppose that there exists a constant* C *such that*

$$\forall \mathbf{v} \in \mathbf{X}, \quad \forall \alpha \in [0;1], \quad \|\mathbf{v}\|_{\mathbf{X}} \le C\, \|L_\alpha \mathbf{v}\|_{\mathbf{Y}}\,.$$

Then L_1 *is surjective onto* $\mathbf{Y}$ *if and only if* L_0 *is surjective onto* $\mathbf{Y}$*.*

We end this section by listing some inequalities which will be frequently used in the sequel.

- Let $f \in H^1(\Omega)$, then the following Gagliardo-Nirenberg's interpolation inequalities hold (see [32]):

$$\|f\|_{L^4} \le C\, \|f\|_{L^2}^{1/4}\, \|f\|_{H^1}^{3/4}\,,$$
$$\|f\|_{L^3} \le C\, \|f\|_{L^2}^{1/2}\, \|f\|_{H^1}^{1/2}\,.$$

- Let $f \in H^2(\Omega)$, then the Agmon's inequality holds (see [1]):

$$\|f\|_{L^\infty} \le C\, \|f\|_{H^1}^{1/2}\, \|f\|_{H^2}^{1/2}\,.$$

  As a consequence, if $f \in H^1(\Omega)$ and $g \in H^2(\Omega)$,

$$\|\nabla(f\, g)\|_{L^2} \le \|f\, g\|_{H^1} \le C\, \|f\|_{H^1}\, \|g\|_{H^1}^{1/2}\, \|g\|_{H^2}^{1/2}\,. \tag{29}$$

- Let a and b two positive real numbers and $1 \le p, q \le \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then for any $\varepsilon > 0$ the following Young inequality holds:

$$ab \le \varepsilon\, a^p + C_\varepsilon\, b^q \quad \text{with} \quad C_\varepsilon = \frac{1}{\varepsilon^{q/p}\, p^{q/p}\, q}.$$

- Let $1 \le p_1, \dots, p_n \le \infty$ and r defined as $\dfrac{1}{r} = \displaystyle\sum_{k=1}^{n} \dfrac{1}{p_k}$. Let suppose that $f_k \in L^{p_k}(\Omega)$ for $1 \le k \le n$. Then the generalized Hölder inequality holds:

$$\Big|\Big| \prod_{i=1}^{n} f_i \Big|\Big|_{L^r(\Omega)} \le \prod_{i=1}^{n} ||f_i||_{L^{p_i}(\Omega)}.$$

3 Estimates for the approximate solutions

Using the results of the previous section, we are able to prove the following result for the sequence $(\mathbf{v}^n, p^n, \vartheta^n)$.

Theorem 3. *Let us assume that the data satisfy* (17)-(18)-(19)-(20)*. Then, the sequence* $(\mathbf{v}^n, p^n, \vartheta^n)$ *is uniquely defined and one has for all* $n \ge 1$*:*

$$\mathbf{v}^n \in \mathcal{C}(0,T;\boldsymbol{V}) \cap L^2(0,T;D(\mathbb{A})), \quad \partial_t \mathbf{v}^n \in L^2(0,T;\boldsymbol{H}); \tag{30}$$
$$p^n \in L^2(0,T;H^1(\Omega) \cap L^2_0(\Omega)); \tag{31}$$
$$\vartheta^n \in \mathcal{C}(0,T;H^2_N(\Omega)) \cap L^2(0,T;H^3_N(\Omega)), \quad \partial_t \vartheta^n \in L^2(0,T;H^1(\Omega)). \tag{32}$$

Furthermore, for all $n \geq 1$, one has:

$$m \leq \vartheta^n \leq M, \tag{33}$$

and there exist two constants K_1 and K_2 only depending on the domain and the data such that

$$\lambda \left\|\Delta\vartheta^n\right\|^2_{\mathcal{C}(0,T;L^2)} + \lambda^2\, m \left\|\nabla\Delta\vartheta^n\right\|^2_{L^2(0,T;L^2)} + \frac{m}{8\,M^2} \left\|\partial_t\vartheta^n\right\|^2_{L^2(0,T;H^1)} \leq K_2, \tag{34}$$

$$\left\|\sqrt{\mu(\vartheta^n)}\nabla\mathbf{v}^n\right\|_{\mathcal{C}(0,T;\mathbb{L}^2)} + \frac{1}{M} \left\|\partial_t\mathbf{v}^n\right\|_{L^2(0,T;\boldsymbol{H})} + \frac{\lambda\, m^2}{8\,M} \left\|\mathbb{A}\mathbf{v}^n\right\|_{L^2(0,T;\boldsymbol{H})} \leq K_1. \tag{35}$$

The proof of Theorem 3 is done by induction on n. We consequently assume that at iteration $n-1$, we have

$$\vartheta^{n-1} \in \mathcal{C}(0,T;H^2_N(\Omega)), \quad \mathbf{v}^{n-1} \in \mathcal{C}(0,T;\mathbf{V}), \tag{36}$$

$$m \leq \vartheta^{n-1}(t) \leq M \quad \text{a.e.}\, t \in [0,T], \tag{37}$$

$$\lambda \left\|\Delta\vartheta^{n-1}(t)\right\|^2_{L^2} \leq K_2, \quad \text{a.e.}\, t \in [0,T], \tag{38}$$

$$\left\|\sqrt{\mu(\vartheta^{n-1})}\nabla\mathbf{v}^{n-1}(t)\right\|^2_{L^2} \leq K_1 \quad \text{a.e.}\, t \in [0,T]. \tag{39}$$

Then, properties (30) to (35) will be established at iteration n. Consequently, properties (36) to (39) will also occur at iteration n. Several steps have to be followed. First, the existence and uniqueness of the temperature ϑ^n solution of (40) are established in subsection 3.1 (Lemma 7), as well as some *a priori* estimates (Lemmas 8 and 9). Then, the existence and uniqueness as well as some *a priori* estimates are proved for the velocity $\mathbf{v}^n$ solution of (43) in subsection 3.2 (Lemma 10). Consequently, the proof of Theorem 3 can be easily obtained in subsection 3.3.

3.1 Existence, uniqueness and a priori estimates for the temperature

Existence and uniqueness We start by showing the existence of the temperature ϑ^n. Note that we will obtain existence only under the smallness induction hypothesis (38) on ϑ^{n-1}.

Lemma 7. *Let us consider the weak formulation of equation* (15) *given by:*

$$\begin{cases} \langle\partial_t\vartheta^n,\eta\rangle_{H^{1\prime},H^1} + \left(\mathbf{v}^{n-1}\cdot\nabla\vartheta^n,\eta\right)_{L^2} + 2\lambda\left(\nabla\vartheta^{n-1}\cdot\nabla\vartheta^n,\eta\right)_{L^2} \\ +\lambda\left(\vartheta^{n-1}\nabla\vartheta^n,\nabla\eta\right)_{L^2} = 0, \quad a.e.\, t \in [0,T], \quad \forall\eta \in H^1(\Omega), \\ \vartheta^n(0,\cdot) = \vartheta_0. \end{cases} \tag{40}$$

Under induction assumptions (36)–(37)–(38)*, the system* (40) *admits a unique solution* $\vartheta^n \in \mathcal{C}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ *such that* $\partial_t\vartheta^n \in L^2(0,T;H^1(\Omega)')$.

Proof. For a.e. $t \in [0,T]$ and $\forall\vartheta,\eta \in H^1(\Omega)$, we set:

$$a(t,\vartheta,\eta) = \int_\Omega \eta\,\mathbf{v}^{n-1}\cdot\nabla\vartheta + 2\lambda\int_\Omega \eta\,\nabla\vartheta^{n-1}\cdot\nabla\vartheta + \lambda\int_\Omega \vartheta^{n-1}\nabla\vartheta\cdot\nabla\eta. \tag{41}$$

Then we consider the right-hand-side of (41). The first integral is zero because $\mathbf{v}^{n-1} \in \mathbf{V}$. Thanks to (37), the second one verifies

$$\lambda\int_\Omega \vartheta^{n-1}|\nabla\vartheta|^2 \geq \lambda\, m \left\|\nabla\vartheta\right\|^2_{L^2}.$$

For the third integral, we use Sobolev injections and a Gagliardo-Nirenberg inequality to obtain:

$$\begin{aligned}\left|2\lambda\int_\Omega \vartheta\,\nabla\vartheta^{n-1}\cdot\nabla\vartheta\right| &\le C\,\lambda\left\|\Delta\vartheta^{n-1}\right\|_{L^2}\|\vartheta\|_{H^1}^{3/2}\|\vartheta\|_{L^2}^{1/2}\\ &\le \frac{\lambda\,m}{2}\|\vartheta\|_{H^1}^2+\frac{C\,\lambda}{m^3}\left\|\Delta\vartheta^{n-1}\right\|_{L^2}^4\|\vartheta\|_{L^2}^2\,.\end{aligned}$$

Thanks to (38), we get:

$$\begin{aligned}a(t,\vartheta,\vartheta) &\ge \lambda\,m\,\|\nabla\vartheta\|_{L^2}^2-\frac{\lambda\,m}{2}\|\vartheta\|_{H^1}^2-\frac{C\,\lambda}{m^3}\left\|\Delta\vartheta^{n-1}\right\|_{L^2}^4\|\vartheta\|_{L^2}^2\\ &\ge \frac{\lambda\,m}{2}\|\vartheta\|_{H^1}^2-\left(\lambda\,m+\frac{C}{\lambda\,m^3}K_2^2\right)\|\vartheta\|_{L^2}^2\,.\end{aligned}$$

Lemma 4 allows to conclude. □

Maximum principle. We now check the maximum principle for ϑ^n. Again, the induction hypothesis of smallness of ϑ^{n-1} is crucial.

Lemma 8. *Let us assume that ϑ_0 satisfies* (18)*. Under the smallness induction hypotheses* (36)–(37)-(38)*, ϑ^n verifies* (33)*.*

Proof. The equation satisfied par $m-\vartheta^n$ is

$$\begin{aligned}&<\partial_t(m-\vartheta^n),\eta>_{H^{1\prime},H^1}+\int_\Omega\eta\,\mathbf{v}^{n-1}\cdot\nabla(m-\vartheta^n)+2\lambda\int_\Omega\eta\,\nabla\vartheta^{n-1}\cdot\nabla(m-\vartheta^n)\\ &+\lambda\int_\Omega\vartheta^{n-1}\nabla(m-\vartheta^n)\cdot\nabla\eta=0,\quad\forall\eta\in H^1(\Omega).\end{aligned}$$

We denote by $\vartheta^n_-=\max(0,m-\vartheta^n)$ and choose $\eta=\vartheta^n_-$. We get:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vartheta^n_-\right\|_{L^2}^2+\int_\Omega\vartheta^n_-\,\mathbf{v}^{n-1}\cdot\nabla\vartheta^n_-+2\lambda\int_\Omega\vartheta^n_-\,\nabla\vartheta^{n-1}\cdot\nabla\vartheta^n_-+\lambda\int_\Omega\vartheta^{n-1}|\nabla\vartheta^n_-|^2=0.$$

The same arguments as those used in the proof of Lemma 7 show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vartheta^n_-\right\|_{L^2}^2+\frac{\lambda\,m}{2}\left\|\nabla\vartheta^n_-\right\|_{L^2}^2\le\left(\frac{C}{\lambda\,m^3}K_2^2+\frac{\lambda\,m}{2}\right)\left\|\vartheta^n_-\right\|_{L^2}^2\,.$$

Then, Gronwall's Lemma gives

$$\left\|\vartheta^n_-(t)\right\|_{L^2}^2+\lambda\,m\int_0^t\left\|\nabla\vartheta^n_-\right\|_{L^2}^2\le\left\|\vartheta^n_-(0)\right\|_{L^2}^2\exp\left(\left(\frac{C}{\lambda\,m^3}K_2^2+\lambda\,m\right)T\right)\qquad\text{a.e.}\,t\in[0,T].$$

Since $\vartheta^n_-(0)=0$, we deduce that $\vartheta^n(t)\ge m$ a.e. $t\in[0,T]$. The upper bound is obtained in a similar way. □

A priori estimates We now give some *a priori* estimates for ϑ^n that will allow us to pass to the limit.

Lemma 9. *Under the induction hypotheses* (36)-(37)-(38)-(39)*, there exists a constant $C_1>0$ only depending on the domain such that*

$$\begin{aligned}&\lambda\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\vartheta^n\|_{L^2}^2+\lambda^2\,m\,\|\nabla\Delta\vartheta^n\|_{L^2}^2+\frac{m}{8\,M^2}\|\partial_t\vartheta^n\|_{H^1}^2\\ \le&\frac{C_1}{\lambda^3\,m^3}\left(\left\|\nabla\mathbf{v}^{n-1}\right\|_{L^2}^4+\lambda^4\left\|\Delta\vartheta^{n-1}\right\|_{L^2}^4+\lambda^4\,M^4\right)\lambda\,\|\Delta\vartheta^n\|_{L^2}^2\,.\end{aligned}\tag{42}$$

Proof. Let $\delta > 0$ be a parameter introduced to conveniently balance the estimates. We multiply (15) by $\frac{\delta}{m}\partial_t\vartheta^n$ to get a first equation. Then we take the gradient of (15) and multiply it by $\frac{\delta}{m}\nabla\partial_t\vartheta^n - \lambda\nabla\Delta\vartheta^n$ to get a second equation. Summing these two equations and integrating by parts (the boundary terms vanish because $\vartheta^n \in H^2_N(\Omega)$), we get

$$
\begin{aligned}
&\lambda \int_\Omega \Delta\vartheta^n\, \partial_t\Delta\vartheta^n + \lambda^2 \int_\Omega \vartheta^{n-1}|\nabla\Delta\vartheta^n|^2 + \frac{\delta}{m}\int_\Omega |\partial_t\vartheta^n|^2 + \frac{\delta}{m}\int_\Omega |\nabla\partial_t\vartheta^n|^2 \\
=&\lambda \int_\Omega \nabla\Delta\vartheta^n \cdot \nabla(\mathbf{v}^{n-1}\cdot\nabla\vartheta^n) + \lambda^2 \int_\Omega \nabla\Delta\vartheta^n \cdot \nabla(\nabla\vartheta^{n-1}\cdot\nabla\vartheta^n) \\
&- \lambda^2 \int_\Omega (\nabla\Delta\vartheta^n\cdot\nabla\vartheta^{n-1})\Delta\vartheta^n + \frac{\delta}{m}\int_\Omega \partial_t\vartheta^n\, \mathbf{v}^{n-1}\cdot\nabla\vartheta^n + \frac{\delta\,\lambda}{m}\int_\Omega \partial_t\vartheta^n\,\nabla\vartheta^{n-1}\cdot\nabla\vartheta^n \\
&- \frac{\delta\,\lambda}{m}\int_\Omega (\partial_t\vartheta^n\,\vartheta^{n-1})\Delta\vartheta^n + \frac{\delta}{m}\int_\Omega \nabla\partial_t\vartheta^n\cdot\nabla(\mathbf{v}^{n-1}\cdot\nabla\vartheta^n) \\
&+ \frac{\delta\,\lambda}{m}\int_\Omega \nabla\partial_t\vartheta^n\cdot\nabla(\nabla\vartheta^{n-1}\cdot\nabla\vartheta^n) - \frac{\delta\,\lambda}{m}\int_\Omega (\nabla\partial_t\vartheta^n\cdot\nabla\vartheta^{n-1})\Delta\vartheta^n \\
&- \frac{\delta\,\lambda}{m}\int_\Omega \vartheta^{n-1}\,\nabla\partial_t\vartheta^n\cdot\nabla\Delta\vartheta^n.
\end{aligned}
$$

Thank's to Hölder and Young inequalities, we obtain

$$
\begin{aligned}
&\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\vartheta^n\|^2_{L^2} + \lambda^2\, m\Big(\frac{7}{8} - \frac{2\,\delta\,M^2}{m^2}\Big)\|\nabla\Delta\vartheta^n\|^2_{L^2} + \frac{\delta}{2m}\|\partial_t\vartheta^n\|^2_{H^1} \\
\le& \frac{C_\delta}{m}\Big(\left\|\nabla(\mathbf{v}^{n-1}\cdot\nabla\vartheta^n)\right\|^2_{L^2} + \lambda^2\left\|\nabla(\nabla\vartheta^{n-1}\cdot\nabla\vartheta^n)\right\|^2_{L^2} + \lambda^2\left\|\nabla\vartheta^{n-1}\right\|^2_{L^6}\|\Delta\vartheta^n\|^2_{L^3}\Big) \\
&+ \frac{C_\delta}{m}\Big(\left\|\mathbf{v}^{n-1}\right\|^2_{L^4}\|\nabla\vartheta^n\|^2_{L^4} + \lambda^2\left\|\nabla\vartheta^{n-1}\right\|^2_{L^4}\|\nabla\vartheta^n\|^2_{L^4} + \lambda^2\left\|\vartheta^{n-1}\right\|^2_{L^\infty}\|\Delta\vartheta^n\|^2_{L^2}\Big).
\end{aligned}
$$

We now use Gagliardo-Nirenberg's inequality and (29) associated to the rough estimate

$$
\|\nabla\vartheta^n\|^2_{L^4} \le C\,\|\Delta\vartheta^n\|_{L^2}\|\nabla\Delta\vartheta^n\|_{L^2},
$$

to get:

$$
\begin{aligned}
&\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\vartheta^n\|^2_{L^2} + \lambda^2\, m\Big(\frac{7}{8} - \frac{2\,\delta\,M^2}{m^2}\Big)\|\nabla\Delta\vartheta^n\|^2_{L^2} + \frac{\delta}{2m}\|\partial_t\vartheta^n\|^2_{H^1} \\
\le& \frac{C_\delta}{m}\Big(\left\|\nabla\mathbf{v}^{n-1}\right\|^2_{L^2}\|\Delta\vartheta^n\|_{L^2}\|\nabla\Delta\vartheta^n\|_{L^2} + \lambda^2\left\|\Delta\vartheta^{n-1}\right\|^2_{L^2}\|\Delta\vartheta^n\|_{L^2}\|\nabla\Delta\vartheta^n\|_{L^2} \\
&\qquad +\lambda^2 M^2\|\Delta\vartheta^n\|_{L^2}\|\nabla\Delta\vartheta^n\|_{L^2}\Big).
\end{aligned}
$$

Using Young's inequality yields:

$$
\begin{aligned}
&\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\vartheta^n\|^2_{L^2} + \lambda^2\, m\Big(\frac{3}{4} - \frac{2\,\delta\,M^2}{m^2}\Big)\|\nabla\Delta\vartheta^n\|^2_{L^2} + \frac{\delta}{2m}\|\partial_t\vartheta^n\|^2_{H^1} \\
\le& \frac{C_\delta}{m^3\,\lambda^2}\Big(\left\|\nabla\mathbf{v}^{n-1}\right\|^4_{L^2}\|\Delta\vartheta^n\|^2_{L^2} + \lambda^4\left\|\Delta\vartheta^{n-1}\right\|^4_{L^2}\|\Delta\vartheta^n\|^2_{L^2} + \lambda^4 M^4\|\Delta\vartheta^n\|^2_{L^2}\Big).
\end{aligned}
$$

We finally choose $\delta = \dfrac{m^2}{8M^2}(< 1)$ to obtain (42).

□

3.2 Existence, uniqueness and a priori estimates for the velocity

To our knowledge, there is no existence result of a solution to the equation (16) in the literature. We therefore follow the method of [6] to establish the existence of a solution in addition to *a priori* estimates. We are looking for $\mathbf{v}^n \in L^2(0,T;D(\mathbb{A})) \cap L^\infty(0,T,\mathbf{V})$ such that $\partial_t \mathbf{v}^n \in L^2(0,T;\mathbf{H})$ and satisfying

$$\begin{cases} \text{a.e. } t \in [0,T], \quad \forall \mathbf{w} \in \mathbf{V}, \\ \left(\frac{1}{\vartheta^n}\partial_t \mathbf{v}^n, \mathbf{w}\right)_{L^2} + b\left(\frac{\mathbf{v}^{n-1}}{\vartheta^n}, \mathbf{v}^n, \mathbf{w}\right) + (\mu(\vartheta^n)\nabla \mathbf{v}^n, \nabla \mathbf{w})_{L^2} \\ +\lambda\, c\left(\mathbf{v}^n, \vartheta^n, \frac{\mathbf{w}}{\vartheta^n}\right) = \left(\frac{\mathbf{f}}{\vartheta^n}, \mathbf{w}\right)_{L^2}, \\ \mathbf{v}^n(0,\cdot) = \mathbf{v}_0, \end{cases} \tag{43}$$

with

$$\begin{aligned} b(\mathbf{u},\mathbf{v},\mathbf{w}) &= \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} \\ c(\mathbf{v},\vartheta,\frac{\mathbf{w}}{\vartheta}) &= \int_\Omega \frac{1}{\vartheta}\left(\nabla\mathbf{v} - \nabla^t\mathbf{v}\right)\nabla\vartheta\cdot\mathbf{w}, \end{aligned}$$

where $\vartheta^n \in \mathcal{C}(0,T;H^2_N(\Omega)) \cap L^2(0,T;H^3_N(\Omega))$ is the solution of (40) and especially satisfies (33).

Lemma 10. *Under the induction hypotheses* (36)-(37)-(38)-(39)*, equation* (43) *admits a unique solution* $\mathbf{v}^n$ *such that*

$$\mathbf{v}^n \in \mathcal{C}(0,T;\boldsymbol{V}) \cap L^2(0,T;D(\mathbb{A})) \quad ; \quad \partial_t \mathbf{v}^n \in L^2(0,T,\boldsymbol{H}).$$

Furthermore, $\mathbf{v}^n$ *is bounded in these spaces and there exists a positive constant* C_2 *such that:*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu(\vartheta^n)|\nabla\mathbf{v}^n|^2 + \frac{1}{M}\|\partial_t\mathbf{v}^n\|^2_{L^2} + \frac{m^2\,\lambda^2\,|\ln M|^2}{8\,M}\|\mathbb{A}\mathbf{v}^n\|^2_{L^2} \\ \leq C_2 &\left\{ \frac{M^4\left(\lambda^2\,|\ln M|^4\,m^4 + K_2^2\right)}{\lambda^7\,m^{11}\,\min\left(|\ln M|^7, |\ln M|^{11}\right)}\left(|\ln M|^{-2}K_1^2 + \lambda^4\,K_2^2\right)\int_\Omega \mu(\vartheta^n)|\nabla\mathbf{v}^n|^2 \right. \\ &\left. + \frac{M}{m^2}\|\mathbf{f}\|^2_{L^2} \right\}. \end{aligned} \tag{44}$$

Proof. We will apply the continuity method. We first set:

$$\begin{aligned} \mathbf{X} &= \left\{\mathbf{v} \in L^2(0,T;D(\mathbb{A}));\ \partial_t\mathbf{v} \in L^2(0,T,\mathbf{H})\right\}, \\ \mathbf{Y} &= L^2(0,T;\mathbf{H}) \times \mathbf{V}, \end{aligned}$$

equipped with the following norms:

$$\|\mathbf{v}\|^2_{\mathbf{X}} = \left\|\sqrt{\mu(\vartheta^n)}\nabla\mathbf{v}\right\|^2_{\mathcal{C}(0,T;\mathbb{L}^2)} + \frac{1}{M}\|\partial_t\mathbf{v}\|^2_{L^2(0,T;\mathbf{H})} + \frac{\lambda^2\,m^2\,|\ln M|^2}{8\,M}\|\mathbb{A}\mathbf{v}\|^2_{L^2(0,T;\mathbf{H})}$$

and

$$\|(\mathbf{f},\mathbf{v}_0)\|^2_{\mathbf{Y}} = \left\|\sqrt{\mu(\vartheta_0)}\nabla\mathbf{v}_0\right\|^2_{\mathbb{L}^2} + \frac{M}{m^2}\|\mathbf{f}\|^2_{L^2(\mathcal{Q}_T)}.$$

It is well known that $\mathbf{X} \hookrightarrow \mathcal{C}(0,T;\mathbf{V})$. We also set

$$\widehat{\vartheta} = \frac{m+M}{2}, \tag{45}$$

and define for $\alpha \in [0,1]$, the operators $L_\alpha : \mathbf{X} \to \mathbf{Y}$ by

$$L_\alpha \mathbf{v} = \left(\mathbb{P}\left(\left(\frac{1-\alpha}{\widehat{\vartheta}} + \frac{\alpha}{\vartheta^n} \right) \partial_t \mathbf{v} + \frac{1}{\vartheta^n}(\mathbf{v}^{n-1}\cdot\nabla)\mathbf{v} - div(\mu(\vartheta^n)\nabla\mathbf{v}) + \frac{\lambda}{\vartheta^n}(\nabla\mathbf{v} - \nabla^t\mathbf{v})\nabla\vartheta^n \right), \mathbf{v}|_{t=0} \right).$$

First of all, we prove that L_0 maps $\mathbf{X}$ onto $\mathbf{Y}$ by using a Fædo-Galerkin approach. For each $N \geq 1$, we set $\mathbf{V}_N =< \mathbf{w}_1, \cdots, \mathbf{w}_N >$ and consider $\mathbf{v}_N$ the unique solution of

$$\begin{cases} \text{a.e.}\, t \in [0,T], \quad \forall \mathbf{w} \in \mathbf{V}_N, \\ \left(\frac{1}{\widehat{\vartheta}} \partial_t \mathbf{v}_N, \mathbf{w} \right)_{L^2} + b\left(\frac{\mathbf{v}^{n-1}}{\vartheta^n}, \mathbf{v}_N, \mathbf{w} \right) + (\mu(\vartheta^n)\nabla\mathbf{v}_N, \nabla\mathbf{w})_{L^2} \\ +\lambda\, c\left(\mathbf{v}_N, \vartheta^n, \frac{\mathbf{w}}{\vartheta^n} \right) = \left(\frac{\mathbf{f}}{\vartheta^n}, \mathbf{w} \right)_{L^2}, \\ \mathbf{v}_N(0,\cdot) = \mathbf{v}_0. \end{cases} \tag{46}$$

To show that the sequence $(\mathbf{v}_N)$ converges toward $\mathbf{v}^n$ as N goes to infinity, we will prove an *a priori* estimate by choosing $\mathbf{w} = \partial_t \mathbf{v}_N + \delta\, \mathbb{A}\mathbf{v}_N$ in (46). As in the proof of Lemma 9, the positive parameter δ will allow us to balance the estimates. We thus obtain:

$$\begin{aligned} &\frac{1}{\widehat{\vartheta}} \|\partial_t \mathbf{v}_N\|_{L^2}^2 + \frac{\delta}{\widehat{\vartheta}} \int_\Omega \partial_t \mathbf{v}_N \cdot \mathbb{A}\mathbf{v}_N + \int_\Omega \frac{1}{\vartheta^n}(\mathbf{v}^{n-1}\cdot\nabla)\mathbf{v}_N \cdot \partial_t \mathbf{v}_N \\ &+ \delta \int_\Omega \frac{1}{\vartheta^n}\,(\mathbf{v}^{n-1}\cdot\nabla)\mathbf{v}_N \cdot \mathbb{A}\mathbf{v}_N + \int_\Omega \mu(\vartheta^n)\nabla\mathbf{v}_N \cdot \nabla\partial_t\mathbf{v}_N - \delta \int_\Omega \operatorname{div}\left(\mu(\vartheta^n)\nabla\mathbf{v}_N\right) \cdot \mathbb{A}\mathbf{v}_N \\ &+ \lambda \int_\Omega \frac{1}{\vartheta^n}(\nabla\mathbf{v}_N - \nabla^t\mathbf{v}_N)\nabla\vartheta^n \cdot \partial_t\mathbf{v}_N + \delta\lambda \int_\Omega \frac{1}{\vartheta^n}\,(\nabla\mathbf{v}_N - \nabla^t\mathbf{v}_N)\nabla\vartheta^n \cdot \mathbb{A}\mathbf{v}_N \\ = &\int_\Omega \frac{\mathbf{f}}{\vartheta^n} \cdot \partial_t \mathbf{v}_N + \delta \int_\Omega \frac{\mathbf{f}}{\vartheta^n} \cdot \mathbb{A}\mathbf{v}_N. \end{aligned}$$

On the one hand, we observe that with (40):

$$\begin{aligned} \int_\Omega \mu(\vartheta^n)\nabla\mathbf{v}_N \cdot \nabla\partial_t\mathbf{v}_N = &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mu(\vartheta^n)|\nabla\mathbf{v}_N|^2 + \frac{1}{2}\int_\Omega \mu'(\vartheta^n)\mathbf{v}^{n-1}\cdot\nabla\vartheta^n|\nabla\mathbf{v}_N|^2 \\ &+ \frac{\lambda}{2}\int_\Omega \mu'(\vartheta^n)\nabla\vartheta^{n-1}\cdot\nabla\vartheta^n|\nabla\mathbf{v}_N|^2 \\ &- \frac{\lambda}{2}\int_\Omega \mu'(\vartheta^n)\vartheta^{n-1}\Delta\vartheta^n|\nabla\mathbf{v}_N|^2 \end{aligned}$$

On the other hand, using the Helmholtz decomposition (27), we have:

$$\begin{aligned} -\int_\Omega \operatorname{div}\left(\mu(\vartheta^n)\nabla\mathbf{v}_N\right) \cdot \mathbb{A}\mathbf{v}_N = &\int_\Omega \mu(\vartheta^n)\mathbb{A}\mathbf{v}_N \cdot \mathbb{A}\mathbf{v}_N + \int_\Omega \mu(\vartheta^n)\nabla q_N \cdot \mathbb{A}\mathbf{v}_N \\ &- \int_\Omega \mu'(\vartheta^n)\,(\nabla\vartheta^n\cdot\nabla)\mathbf{v}_N \cdot \mathbb{A}\mathbf{v}_N. \end{aligned}$$

Note that because $\mathbb{A}\mathbf{v}_N \in \mathbf{V}$, an integration by parts gives:

$$\int_\Omega \mu(\vartheta^n)\nabla q_N \cdot \mathbb{A}\mathbf{v}_N = -\int_\Omega \mu'(\vartheta^n)\, q_N\, \nabla\vartheta^n \cdot \mathbb{A}\mathbf{v}_N.$$

Then, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu(\vartheta^n)|\nabla \mathbf{v}_N|^2 + \frac{1}{\widehat{\vartheta}}\,\|\partial_t \mathbf{v}_N\|_{L^2}^2 + \delta\int_\Omega \mu(\vartheta^n)|\mathbb{A}\mathbf{v}_N|^2\\
=&-\frac{\delta}{\widehat{\vartheta}}\int_\Omega \partial_t\mathbf{v}_N\cdot\mathbb{A}\mathbf{v}_N - \int_\Omega \frac{1}{\vartheta^n}(\mathbf{v}^{n-1}\cdot\nabla)\mathbf{v}_N\cdot\partial_t\mathbf{v}_N - \delta\int_\Omega \frac{1}{\vartheta^n}\,(\mathbf{v}^{n-1}\cdot\nabla)\mathbf{v}_N\cdot\mathbb{A}\mathbf{v}_N\\
&-\frac{1}{2}\int_\Omega \mu'(\vartheta^n)\mathbf{v}^{n-1}\cdot\nabla\vartheta^n|\nabla\mathbf{v}_N|^2 - \frac{\lambda}{2}\int_\Omega \mu'(\vartheta^n)\nabla\vartheta^{n-1}\cdot\nabla\vartheta^n|\nabla\mathbf{v}_N|^2\\
&+\frac{\lambda}{2}\int_\Omega \mu'(\vartheta^n)\vartheta^{n-1}\Delta\vartheta^n|\nabla\mathbf{v}_N|^2 + \delta\int_\Omega \mu'(\vartheta^n)\,q_N\,\nabla\vartheta^n\cdot\mathbb{A}\mathbf{v}_N\\
&+\delta\int_\Omega \mu'(\vartheta^n)\,(\nabla\vartheta^n\cdot\nabla)\mathbf{v}_N\cdot\mathbb{A}\mathbf{v}_N - \lambda\int_\Omega \frac{1}{\vartheta^n}\,(\nabla\mathbf{v}_N - \nabla^t\mathbf{v}_N)\nabla\vartheta^n\cdot\partial_t\mathbf{v}_N\\
&-\delta\lambda\int_\Omega \frac{1}{\vartheta^n}\,(\nabla\mathbf{v}_N - \nabla^t\mathbf{v}_N)\nabla\vartheta^n\cdot\mathbb{A}\mathbf{v}_N + \int_\Omega \frac{\mathbf{f}}{\vartheta^n}\cdot\partial_t\mathbf{v}_N + \delta\int_\Omega \frac{\mathbf{f}}{\vartheta^n}\cdot\mathbb{A}\mathbf{v}_N.
\end{aligned}
$$

To obtain the wanted estimates, we notice that with the particular choice of the viscosity (6), we have:

$$
\begin{aligned}
\lambda|\ln M| \le \mu(\vartheta^n) \le \lambda|\ln m|,\\
\frac{\lambda}{M} \le |\mu'(\vartheta^n)| \le \frac{\lambda}{m}.
\end{aligned}
$$

We detail in particular the estimate of $\delta\int_\Omega \mu'(\vartheta^n)\,q_N\,\nabla\vartheta^n\cdot\mathbb{A}\mathbf{v}_N$, the other ones are classical and can be obtained involving Holder, Gagliardo-Nirenberg, Sobolev and Young inequalities. Let $\varepsilon > 0$ be a small parameter and note that $\|\nabla q\|_{L^2} \le C\,\|\mathbb{A}\mathbf{v}\|_{L^2}$. Using Lemma 5, we have:

$$
\begin{aligned}
\left|\delta\int_\Omega \mu'(\vartheta^n)\,q_N\,\nabla\vartheta^n\cdot\mathbb{A}\mathbf{v}_N\right| \le&\,\delta\frac{C\,\lambda}{m}\,\|q_N\|_{L^2}^{1/2}\,\|\nabla q_N\|_{L^2}^{1/2}\,\|\Delta\vartheta^n\|_{L^2}\,\|\mathbb{A}\mathbf{v}_N\|_{L^2}\\
\le&\,\delta\,\varepsilon\,\lambda|\ln M|\,\|\mathbb{A}\mathbf{v}_N\|_{L^2}^2 + \delta\frac{C_\varepsilon\,\lambda}{m^4\,|\ln M|^3}\,\|\Delta\vartheta^n\|_{L^2}^4\,\|q_N\|_{L^2}^2\\
\le&\,\delta\,\varepsilon\,\lambda|\ln M|\,\|\mathbb{A}\mathbf{v}_N\|_{L^2}^2 + \frac{C_\varepsilon\,\delta\,\lambda\,\|\Delta\vartheta^n\|_{L^2}^4}{m^4\,|\ln M|^3}\\
&\times\left(\frac{\varepsilon\,|\ln M|^4\,m^4}{\|\Delta\vartheta^n\|_{L^2}^4}\,\|\mathbb{A}\mathbf{v}_N\|_{L^2}^2 + C\left(1+\frac{\|\Delta\vartheta^n\|_{L^2}^4}{\varepsilon\,|\ln M|^4\,m^4}\right)\|\nabla\mathbf{v}_N\|_{L^2}^2\right)\\
\le&\,2\,\varepsilon\,\delta\lambda|\ln M|\,\|\mathbb{A}\mathbf{v}_N\|_{L^2}^2\\
&+\frac{C_\varepsilon\,\delta\,\lambda\,\|\Delta\vartheta^n\|_{L^2}^4}{m^4\,|\ln M|^3}\left(1+\frac{\|\Delta\vartheta^n\|_{L^2}^4}{|\ln M|^4\,m^4}\right)\|\nabla\mathbf{v}_N\|_{L^2}^2\,. \qquad (47)
\end{aligned}
$$

Thus for ε small enough, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu(\vartheta^n)|\nabla \mathbf{v}_N|^2 + \frac{1}{2\widehat{\vartheta}}\|\partial_t \mathbf{v}_N\|_{L^2}^2 + \delta\,\lambda|\ln M|\left(\frac{3}{4}-\frac{2\,\delta}{\widehat{\vartheta}\,\lambda\,|\ln M|}\right)\|\mathbb{A}\mathbf{v}_N\|_{L^2}^2\\
\le C\Bigg\{ &\left(\frac{\widehat{\vartheta}^2}{\delta\, m^4\,\lambda|\ln M|}+\frac{\delta}{\lambda^3|\ln M|^3\, m^4}+\frac{1}{\lambda\,\delta\,|\ln M|\, m^2}\right)\left\|\nabla \mathbf{v}^{n-1}\right\|_{L^2}^4\\
&+\Bigg(\frac{\lambda^3}{\delta\, m^2\,|\ln M|}+\frac{\lambda^5\, M^4}{\delta^3\, m^4\,|\ln M|^3}\\
&\quad+\frac{\delta\,\lambda}{|\ln M|^3\, m^4}\left(1+\frac{\|\Delta\vartheta^n\|_{L^2}^4}{|\ln M|^4\, m^4}\right)+\frac{\lambda^3\,\widehat{\vartheta}^2}{\delta|\ln M|\, m^4}\Bigg)\|\Delta\vartheta^n\|_{L^2}^4\\
&+\frac{\lambda^3}{\delta\, m^2\,|\ln M|}\left\|\Delta\vartheta^{n-1}\right\|_{L^2}^4\|\nabla \mathbf{v}_N\|_{L^2}^2\\
&+\left(\frac{\widehat{\vartheta}}{m^2}+\frac{\delta}{\lambda\,|\ln M|\, m^2}\right)\|\mathbf{f}\|_{L^2}^2\Bigg\}.
\end{aligned}
$$

We now choose $\delta = \frac{\lambda\,|\ln M|\,\widehat{\vartheta}}{8}$. In order to reorder the terms and to have a simplified expression, thanks to the fact that $m \le \widehat{\vartheta} \le M$, we have:

$$
\frac{1}{m^2\,\widehat{\vartheta}} \le \frac{\widehat{\vartheta}}{m^4} \le \frac{M^4}{m^4\,\widehat{\vartheta}^3} \le \frac{M^4}{m^7}.
$$

Then, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu(\vartheta^n)|\nabla \mathbf{v}_N|^2 + \frac{1}{2\,M}\|\partial_t \mathbf{v}_N\|_{L^2}^2 + \frac{\lambda^2\, m\,|\ln M|^2}{16}\|\mathbb{A}\mathbf{v}_N\|_{L^2}^2\\
\le&\frac{C\, M^4}{\lambda^2\, m^7\, \min(|\ln M|^2;|\ln M|^6)}\left(1+\frac{\|\Delta\vartheta^n\|_{L^2}^4}{|\ln M|^4\, m^4}\right)\\
&\times\left(\left\|\nabla \mathbf{v}^{n-1}\right\|_{L^2}^4+\lambda^4\,\|\Delta\vartheta^n\|_{L^2}^4+\lambda^4\left\|\Delta\vartheta^{n-1}\right\|_{L^2}^4\right)\|\nabla \mathbf{v}_N\|_{L^2}^2\\
&+\frac{C\, M}{m^2}\|\mathbf{f}\|_{L^2}^2\,.
\end{aligned}
$$

With the induction hypotheses (38) and (39) and the previous estimate (34), by noting that $\|\nabla \mathbf{v}_N\|_{L^2}^2 \le \frac{1}{\lambda\,|\ln M|}\int_\Omega \mu(\vartheta^n)|\nabla \mathbf{v}_N|^2$, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu(\vartheta^n)|\nabla \mathbf{v}_N|^2 + \frac{1}{2\,M}\|\partial_t \mathbf{v}_N\|_{L^2}^2 + \frac{\lambda^2\, m\,|\ln M|^2}{16}\|\mathbb{A}\mathbf{v}_N\|_{L^2}^2\\
\le&\frac{C\, M^4\left(\lambda^2\,|\ln M|^4\, m^4+K_2^2\right)}{\lambda^7\, m^{11}\,\min\left(|\ln M|^7,|\ln M|^{11}\right)}\left(|\ln M|^{-2}K_1^2+\lambda^4\, K_2^2\right)\int_\Omega \mu(\vartheta^n)|\nabla \mathbf{v}_N|^2\\
&+\frac{C\, M}{m^2}\|\mathbf{f}\|_{L^2}^2\,.
\end{aligned}
$$

Finally, we deduce from Gronwall's Lemma that $(\mathbf{v}_N)$ is bounded in $L^2(0,T;D(\mathbb{A}))\cap L^\infty(0,T;\mathbf{V})$ and that $(\partial_t \mathbf{v}_N)$ is bounded in $L^2(0,T;\mathbf{H})$. Then, standard limit arguments prove that $(\mathbf{v}_N)$ converges towards $\mathbf{v}$, the solution of the problem $L_0\mathbf{v} = (\mathbf{f},\mathbf{v}_0)$. This shows that L_0 is surjective onto $\mathbf{Y}$.

Now, we consider $\mathbf{v} \in \mathbf{X}$ and $\alpha \in [0,1]$. We multiply the equation of the Cauchy problem $L_\alpha \mathbf{v} = (\mathbf{f}, \mathbf{v}_0)$ by $\partial_t \mathbf{v} + \delta \mathbb{A}\mathbf{v}$. The same calculations as those made previously allow us to obtain:

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu(\vartheta^n)|\nabla \mathbf{v}|^2 + \frac{1}{2\,M}\,\|\partial_t \mathbf{v}\|_{L^2}^2 + \frac{m^2\,\lambda^2\,|\ln M|^2}{16\,M}\,\|\mathbb{A}\mathbf{v}\|_{L^2}^2 \\
\leq\ & C\frac{M^4\left(\lambda^2\,|\ln M|^4\,m^4 + K_2^2\right)}{\lambda^7\,m^{11}\,\min\left(|\ln M|^7, |\ln M|^{11}\right)}\left(|\ln M|^{-2}K_1^2 + \lambda^4\,K_2^2\right)\int_\Omega \mu(\vartheta^n)|\nabla \mathbf{v}^n|^2 \\
&+ \frac{C\,M}{m^2}\,\|\mathbf{f}\|_{L^2}^2\,.
\end{aligned}
\tag{48}
$$

We now apply Gronwall's Lemma and get:

$$
\|\mathbf{v}\|_{\mathbf{X}}^2 \leq \exp\left(C_2\,T\,\frac{M^4\left(\lambda^2\,|\ln M|^4\,m^4 + K_2^2\right)}{\lambda^7\,m^{11}\,\min\left(|\ln M|^7, |\ln M|^{11}\right)}\left(|\ln M|^{-2}K_1^2 + \lambda^4\,K_2^2\right)\right) \times \|L_\alpha \mathbf{v}\|_{\mathbf{Y}}^2\,, \tag{49}
$$

where C_2 is a constant only depending on the domain. From Lemma 6, we obtain the existence of the velocity $\mathbf{v}^n$ solution of (43) and the estimate (44). The proof of uniqueness is classical.

□

3.3 Proof of Theorem 3

Using the previous results, we can now easily prove Theorem 3. Lemma 9 implies (32) and Lemma 8 implies (33). Then, Lemma 9 associated to (38) and (39) and Gronwall's Lemma implies (34). Lemma 10 implies (30) and (35). Finally, De Rham's theorem allows to obtain the existence of the pressure as well as (31). The proof is complete.

4 Convergence of the approximate solutions towards the strong solution

We established in section 3 *a priori* estimates on the sequence $(\mathbf{v}^n, p^n, \vartheta^n)$. We now show that it converges towards the unique strong solution $(\mathbf{v}, p, \vartheta)$ of (7)–(8)–(9) and we exhibit convergence rates. We show in particular that $(\mathbf{v}^n, p^n, \vartheta^n)$ is a Cauchy sequence in appropriate spaces. For that, for a generic variable a, we define $a^{(n,s)} = a^{n+s} - a^n$. Starting from (15) and (16), we write the problems satisfied by $\vartheta^{(n,s)}$ and $(\mathbf{v}^{(n,s)}, p^{(n,s)})$:

$$
\begin{cases}
\partial_t \vartheta^{(n,s)} + \mathbf{v}^{(n-1,s)} \cdot \nabla \vartheta^{n+s} + \mathbf{v}^{n-1} \cdot \nabla \vartheta^{(n,s)} - \lambda \vartheta^{(n-1,s)} \Delta \vartheta^{n+s} \\
-\lambda \vartheta^{n-1} \Delta \vartheta^{(n,s)} + \lambda \nabla \vartheta^{(n-1,s)} \cdot \nabla \vartheta^{n+s} + \lambda \nabla \vartheta^{n-1} \cdot \nabla \vartheta^{(n,s)} = 0, \\
\vartheta^{(n,s)}(0,\cdot) = 0,
\end{cases}
\tag{50}
$$

and

$$
\begin{cases}
\dfrac{\partial_t \mathbf{v}^{(n,s)}}{\vartheta^n} - \operatorname{div}\,\left(\mu(\vartheta^n)\nabla \mathbf{v}^{(n,s)}\right) + \nabla p^{(n,s)} = \\
\dfrac{\vartheta^{(n,s)}}{\vartheta^{n+s}\,\vartheta^n}\partial_t \mathbf{v}^{n+s} + \operatorname{div}\,((\mu(\vartheta^{n+s}) - \mu(\vartheta^n))\nabla \mathbf{v}^{n+s}) \\
+\dfrac{\vartheta^{(n,s)}}{\vartheta^{n+s}\,\vartheta^n}(\mathbf{v}^{n+s-1}\cdot\nabla)\mathbf{v}^{n+s} - \dfrac{1}{\vartheta^n}(\mathbf{v}^{(n-1,s)}\cdot\nabla)\mathbf{v}^{n+s} - \dfrac{1}{\vartheta^n}(\mathbf{v}^{n-1}\cdot\nabla)\mathbf{v}^{(n,s)} \\
+\lambda\dfrac{\vartheta^{(n,s)}}{\vartheta^{n+s}\,\vartheta^n}(\nabla \mathbf{v}^{n+s} - \nabla^t \mathbf{v}^{n+s})\,\nabla\vartheta^{n+s} - \dfrac{\lambda}{\vartheta^n}(\nabla \mathbf{v}^{(n,s)} - \nabla^t \mathbf{v}^{(n,s)})\,\nabla\vartheta^{n+s} \\
-\dfrac{\lambda}{\vartheta^n}(\nabla \mathbf{v}^n - \nabla^t \mathbf{v}^n)\,\nabla\vartheta^{(n,s)} - \dfrac{\vartheta^{(n,s)}}{\vartheta^{n+s}\,\vartheta^n}\mathbf{f}, \\
\\
\mathbf{v}^{(n,s)}(0,\cdot) = 0.
\end{cases}
\tag{51}
$$

4.1 Proof of Theorem 1

We first have to prove that:

$$
\left\|\vartheta^{(n,s)}(t)\right\|_{H^1}^2 + \int_0^t \left(\left\|\vartheta^{(n,s)}(s)\right\|_{H^2}^2 + \left\|\partial_t\vartheta^{(n,s)}(s)\right\|_{L^2}^2\right)\mathrm{d}s \le D\left[\frac{(B\,t)^n}{n!}\right]^{\frac{1}{2}},
\tag{52}
$$

$$
\left\|\mathbf{v}^{(n,s)}(t)\right\|_{L^2}^2 + \int_0^t \left\|\mathbf{v}^{(n,s)}(s)\right\|_{H^1}^2\,\mathrm{d}s \le D\left[\frac{(B\,t)^n}{n!}\right]^{\frac{1}{2}}.
\tag{53}
$$

For that, we will obtain *a priori* estimates for the sequences $(\vartheta^{(n,s)})$ and $(\mathbf{v}^{(n,s)})$ from equations (50) and (51) and then apply Gronwall's Lemma with recurrence.

We start by multiplying equation (50) by $\eta = \frac{\delta}{m}\partial_t\vartheta^{(n,s)} + \lambda\vartheta^{(n,s)} - \lambda\Delta\vartheta^{(n,s)}$ with $\delta > 0$. After integration by parts, we have:

$$
\begin{aligned}
&\frac{\delta}{m}\left\|\partial_t\vartheta^{(n,s)}\right\|_{L^2}^2 + \frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vartheta^{(n,s)}\right\|_{L^2}^2 + \frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\vartheta^{(n,s)}\right\|_{L^2}^2 + \lambda^2\int_\Omega \vartheta^{n-1}|\Delta\vartheta^{(n,s)}|^2 \\
=&\,\lambda\int_\Omega \vartheta^{n-1}\,\Delta\vartheta^{(n,s)}\left(\frac{\delta}{m}\partial_t\vartheta^{(n,s)} + \lambda\vartheta^{(n,s)}\right) - \int_\Omega \mathbf{v}^{(n-1,s)}\cdot\nabla\vartheta^{n+s}\,\eta - \int_\Omega \mathbf{v}^{n-1}\cdot\nabla\vartheta^{(n,s)}\,\eta \\
&+\lambda\int_\Omega \vartheta^{(n-1,s)}\,\Delta\vartheta^{n+s}\,\eta - \lambda\int_\Omega \nabla\vartheta^{(n-1,s)}\cdot\nabla\vartheta^{n+s}\,\eta - \lambda\int_\Omega \nabla\vartheta^{n-1}\cdot\nabla\vartheta^{(n,s)}\,\eta.
\end{aligned}
$$

Usual calculations allow to obtain:

$$
\begin{aligned}
&\frac{\delta}{2\,m}\left\|\partial_t\vartheta^{(n,s)}\right\|_{L^2}^2 + \frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vartheta^{(n,s)}\right\|_{H^1}^2 + \lambda^2\,m\left(\frac{3}{4} - \frac{3\,\delta\,M^2}{m^2}\right)\left\|\Delta\vartheta^{(n,s)}\right\|_{L^2}^2 \\
\le&\, C\frac{M^2\,\lambda^2}{m}\left\|\vartheta^{(n,s)}\right\|_{H^1}^2 \\
&+\frac{C_\delta}{m}\Bigg(\left\|\Delta\vartheta^{n+s}\right\|_{L^2}\left\|\nabla\Delta\vartheta^{n+s}\right\|_{L^2}\left\|\mathbf{v}^{(n-1,s)}\right\|_{L^2}^2 + \left\|\mathbb{A}\mathbf{v}^{n-1}\right\|_{L^2}^2\left\|\nabla\vartheta^{(n,s)}\right\|_{L^2}^2 \\
&\qquad+\lambda^2\left\|\Delta\vartheta^{n+s}\right\|_{L^2}\left\|\nabla\Delta\vartheta^{n+s}\right\|_{L^2}\left\|\nabla\vartheta^{(n-1,s)}\right\|_{L^2}^2 \\
&\qquad+\lambda^2\left\|\Delta\vartheta^{n+s}\right\|_{L^2}\left\|\nabla\Delta\vartheta^{n+s}\right\|_{L^2}\left\|\nabla\vartheta^{(n-1,s)}\right\|_{L^2}^2 + \lambda^2\left\|\nabla\Delta\vartheta^{n-1}\right\|_{L^2}^2\left\|\nabla\vartheta^{(n,s)}\right\|_{L^2}^2\Bigg).
\end{aligned}
$$

By choosing $\delta = \frac{m^2}{12M^2}$, we get:

$$
\begin{aligned}
&\frac{m}{24\,M^2}\left\|\partial_t\vartheta^{(n,s)}\right\|_{L^2}^2 + \frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\vartheta^{(n,s)}\right\|_{H^1}^2 + \frac{\lambda^2\,m}{2}\left\|\Delta\vartheta^{(n,s)}\right\|_{L^2}^2 \\
\leq&\frac{C}{m}\left\{\left\|\Delta\vartheta^{n+s}\right\|_{L^2}\left\|\nabla\Delta\vartheta^{n+s}\right\|_{L^2}\left(\left\|\mathbf{v}^{(n-1,s)}\right\|_{L^2}^2 + \lambda^2\left\|\vartheta^{(n-1,s)}\right\|_{H^1}^2\right)\right. \\
&\qquad\left. + \left(\left\|\mathbb{A}\mathbf{v}^{n-1}\right\|_{L^2}^2 + \lambda^2\left\|\nabla\Delta\vartheta^{n-1}\right\|_{L^2}^2 + \lambda^2\,M^2\right)\left\|\vartheta^{(n,s)}\right\|_{H^1}^2\right\}.
\end{aligned}
\tag{54}
$$

Multiplying equation (51) by $\mathbf{v}^{(n,s)}$, we have:

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\mathbf{v}^{(n,s)}}{\sqrt{\vartheta^n}}\right\|_{L^2}^2 + \int_\Omega \mu(\vartheta^n)|\nabla\mathbf{v}^{(n,s)}|^2 \\
=&-\frac{1}{2}\int_\Omega \frac{\partial_t\vartheta^n}{\vartheta^{n2}}|\mathbf{v}^{(n,s)}|^2 + \int_\Omega \frac{\vartheta^{(n,s)}}{\vartheta^{n+s}\,\vartheta^n}\partial_t\mathbf{v}^{n+s}\cdot\mathbf{v}^{(n,s)} - \int_\Omega(\mu(\vartheta^{n+s})-\mu(\vartheta^n))\nabla\mathbf{v}^{n+s}\cdot\nabla\mathbf{v}^{(n,s)} \\
&+\int_\Omega \frac{\vartheta^{(n,s)}}{\vartheta^{n+s}\,\vartheta^n}(\mathbf{v}^{n+s-1}\cdot\nabla)\mathbf{v}^{n+s}\cdot\mathbf{v}^{(n,s)} - \int_\Omega \frac{1}{\vartheta^n}(\mathbf{v}^{(n-1,s)}\cdot\nabla)\mathbf{v}^{n+s}\cdot\mathbf{v}^{(n,s)} \\
&-\int_\Omega \frac{1}{\vartheta^n}(\mathbf{v}^{n-1}\cdot\nabla)\mathbf{v}^{(n,s)}\cdot\mathbf{v}^{(n,s)} + \lambda\int_\Omega \frac{\vartheta^{(n,s)}}{\vartheta^{n+s}\,\vartheta^n}\left(\nabla\mathbf{v}^{n+s} - \nabla^t\mathbf{v}^{n+s}\right)\nabla\vartheta^{n+s}\cdot\mathbf{v}^{(n,s)} \\
&-\lambda\int_\Omega \frac{1}{\vartheta^n}\left(\nabla\mathbf{v}^{(n,s)} - \nabla^t\mathbf{v}^{(n,s)}\right)\nabla\vartheta^{n+s}\cdot\mathbf{v}^{(n,s)} - \lambda\int_\Omega \frac{1}{\vartheta^n}\left(\nabla\mathbf{v}^n - \nabla^t\mathbf{v}^n\right)\nabla\vartheta^{(n,s)}\cdot\mathbf{v}^{(n,s)} \\
&-\int_\Omega \frac{\vartheta^{(n,s)}}{\vartheta^{n+s}\,\vartheta^n}\mathbf{f}\cdot\mathbf{v}^{(n,s)}.
\end{aligned}
$$

The mean value theorem associated with the particular expression of μ give:

$$
\left|\int_\Omega(\mu(\vartheta^{n+s})-\mu(\vartheta^n))\nabla\mathbf{v}^{n+s}\cdot\nabla\mathbf{v}^{(n,s)}\right| \leq \frac{\lambda}{m}\int_\Omega\left|\vartheta^{(n,s)}\,\nabla\mathbf{v}^{n+s}\cdot\nabla\mathbf{v}^{(n,s)}\right|.
$$

Using standard estimations, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\mathbf{v}^{(n,s)}}{\sqrt{\vartheta^n}}\right\|_{L^2}^2 + \frac{\lambda|\ln M|}{2}\left\|\nabla\mathbf{v}^{(n,s)}\right\|_{L^2}^2 \\
\leq&\frac{C}{\lambda\,|\ln M|\,m^4}\left\|\partial_t\vartheta^n\right\|_{H^1}^2\left\|\mathbf{v}^{(n,s)}\right\|_{L^2}^2 + \frac{C}{\lambda\,|\ln M|\,m^4}\left\|\partial_t\mathbf{v}^{n+s}\right\|_{L^2}^2\left\|\vartheta^{(n,s)}\right\|_{H^1}^2 \\
&+\frac{C\,\lambda}{|\ln M|\,m^2}\left\|\mathbb{A}\mathbf{v}^{n+s}\right\|_{L^2}^2\left\|\vartheta^{(n,s)}\right\|_{H^1}^2 \\
&+\frac{C}{\lambda\,m^2}\left\|\nabla\mathbf{v}^{n+s-1}\right\|_{L^2}\left\|\mathbb{A}\mathbf{v}^{n+s}\right\|_{L^2}\left(\left\|\mathbf{v}^{(n,s)}\right\|_{L^2}^2 + \lambda^2\left\|\vartheta^{(n,s)}\right\|_{H^1}^2\right) \\
&+\frac{C}{\lambda\,|\ln M|\,m^2}\left\|\nabla\mathbf{v}^{n+s}\right\|_{L^2}\left\|\mathbb{A}\mathbf{v}^{n+s}\right\|_{L^2}\left\|\mathbf{v}^{(n-1,s)}\right\|_{L^2}^2 + \frac{C}{\lambda^3\,|\ln M|^3\,m^4}\left\|\nabla\mathbf{v}^{n-1}\right\|_{L^2}^4\left\|\mathbf{v}^{(n,s)}\right\|_{L^2}^2 \\
&+\frac{C}{m^2}\left\|\mathbb{A}\mathbf{v}^{n+s}\right\|_{L^2}\left\|\Delta\vartheta^{n+s}\right\|_{L^2}\left(\left\|\mathbf{v}^{(n,s)}\right\|_{L^2}^2 + \lambda^2\left\|\vartheta^{(n,s)}\right\|_{H^1}^2\right) \\
&+\frac{C\,\lambda}{|\ln M|^3\,m^4}\left\|\Delta\vartheta^{n+s}\right\|_{L^2}^4\left\|\mathbf{v}^{(n,s)}\right\|_{L^2}^2 + \frac{C\,\lambda}{|\ln M|\,m^2}\left\|\mathbb{A}\mathbf{v}^n\right\|_{L^2}^2\left\|\vartheta^{(n,s)}\right\|_{H^1}^2 \\
&+\frac{C}{\lambda\,|\ln M|\,m^4}\left\|\mathbf{f}\right\|_{L^2}^2\left\|\vartheta^{(n,s)}\right\|_{H^1}^2.
\end{aligned}
\tag{55}
$$

We now multiply (54) by $2\,\lambda$, (55) by $2\,M$ and we sum. Since

$$\left\|\mathbf{v}^{(n,s)}\right\|_{L^2}^2 \le M \left\|\frac{\mathbf{v}^{(n,s)}}{\sqrt{\vartheta^n}}\right\|_{L^2}^2,$$

we thus obtain

$$a_n'(t) + b_n(t) \le c_n(t)\, a_n(t) + d_n(t)\, a_{n-1}(t)$$

with

$$\begin{aligned}
a_n &= \lambda^2 \left\|\vartheta^{(n,s)}\right\|_{H^1}^2 + M \left\|\frac{\mathbf{v}^{(n,s)}}{\sqrt{\vartheta^n}}\right\|_{L^2}^2, \\
b_n &= \frac{\lambda\, m}{12\, M^2} \left\|\partial_t \vartheta^{(n,s)}\right\|_{L^2}^2 + \lambda^3\, m \left\|\Delta\vartheta^{(n,s)}\right\|_{L^2}^2 + \lambda\, M\, |\ln M| \left\|\nabla \mathbf{v}^{(n,s)}\right\|_{L^2}^2, \\
c_n &= \frac{C}{m}\Bigg\{\frac{1}{\lambda}\left\|\mathbb{A}\mathbf{v}^{n-1}\right\|_{L^2}^2 + \lambda \left\|\nabla\Delta\vartheta^{n-1}\right\|_{L^2}^2 + \lambda\, M^2 + \frac{M}{\lambda\,|\ln M|\, m^3}\left\|\partial_t\vartheta^n\right\|_{H^1}^2 \\
&\quad + \frac{M}{\lambda^3\,|\ln M|\, m^3}\left\|\partial_t \mathbf{v}^{n+s}\right\|_{L^2}^2 + \frac{M}{\lambda\,|\ln M|\, m}\left\|\mathbb{A}\mathbf{v}^{n+s}\right\|_{L^2}^2 \\
&\quad + \frac{M}{\lambda\, m}\left\|\nabla \mathbf{v}^{n+s-1}\right\|_{L^2}\left\|\mathbb{A}\mathbf{v}^{n+s}\right\|_{L^2} + \frac{M}{\lambda^3\,|\ln M|^3\, m^3}\left\|\nabla\mathbf{v}^{n-1}\right\|_{L^2}^4 \\
&\quad + \frac{M}{m}\left\|\mathbb{A}\mathbf{v}^{n+s}\right\|_{L^2}\left\|\Delta\vartheta^{n+s}\right\|_{L^2} + \frac{\lambda\, M}{|\ln M|^3\, m^3}\left\|\Delta\vartheta^{n+s}\right\|_{L^2}^4 \\
&\quad + \frac{M}{\lambda\,|\ln M|\, m}\left\|\mathbb{A}\mathbf{v}^n\right\|_{L^2}^2 + \frac{M}{\lambda^3\,|\ln M|\, m^3}\left\|\mathbf{f}\right\|_{L^2}^2\Bigg\}, \\
d_n &= \frac{C}{m}\left\{\lambda\left\|\Delta\vartheta^{n+s}\right\|_{L^2}\left\|\nabla\Delta\vartheta^{n+s}\right\|_{L^2} + \frac{M}{\lambda\,|\ln M|\, m}\left\|\nabla\mathbf{v}^{n+s}\right\|_{L^2}\left\|\mathbb{A}\mathbf{v}^{n+s}\right\|_{L^2}\right\}.
\end{aligned}$$

Thanks to Therorem 3, (c_n) is bounded in $L^1(0,T)$ and (d_n) is bounded in $L^2(0,T)$. We thus can apply Lemma 3. As $a_n(0) = 0$, we get:

$$a_n(t) + \int_0^t b_n(t) \le D\,\|a_0\|_{L^\infty(0,T)}\left[\frac{(B\,t)^n}{n!}\right]^{1/2} \le D\left[\frac{(B\,t)^n}{n!}\right]^{1/2},$$

with B and D some positive constants, which proves (52) and (53). Consequently, $(\mathbf{v}^n)$ is a Cauchy sequence in $L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$. Thanks to the estimates of Theorem 3, it converges towards $\mathbf{v} \in L^\infty(0,T;\mathbf{V}) \cap L^2(0,T;D(\mathbb{A}))$. Similarly, (ϑ^n) converges towards $\vartheta \in L^\infty(0,T;H^2_N(\Omega) \cap L^2(0,T;H^3_N(\Omega))$. We easily show with a limit argument in (15)–(16) that $(\mathbf{v},p,\vartheta)$ is solution of (7)–(8)–(9) and (12)–(13), p being obtained thanks to De Rham's theorem. The rates given by formulas (21) and (22) are obtained from (52) and (53) by passing to the limit for s (the bounds being uniform with respect to s).

It remains the question of uniqueness. We proceed in a classical way and suppose that $(\mathbf{v}_1,p_1,\vartheta_1)$ and $(\mathbf{v}_2,p_2,\vartheta_2)$ are both solutions of (7)–(8)–(9) and (12)–(13). We note $\vartheta = \vartheta_1 - \vartheta_2$, $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ and $p = p_1 - p_2$. The system of equations satisfied by $(\mathbf{v},p,\vartheta)$ writes:

$$\begin{cases} \partial_t\vartheta + \mathbf{v}_1\cdot\nabla\vartheta + \mathbf{v}\cdot\nabla\vartheta_2 + \lambda\nabla(\vartheta_1+\vartheta_2)\cdot\nabla\vartheta - \lambda\vartheta_1\Delta\vartheta - \lambda\vartheta\Delta\vartheta_2 = 0, \\ \vartheta(0,\cdot) = 0, \qquad \nabla\vartheta\cdot\mathbf{n} = 0, \end{cases} \tag{56}$$

$$
\begin{cases}
\dfrac{1}{\vartheta_1}\partial_t \mathbf{v} - \dfrac{\vartheta}{\vartheta_1\,\vartheta_2}\partial_t \mathbf{v}_2 + \dfrac{1}{\vartheta_1}(\mathbf{v}_1\cdot\nabla)\mathbf{v} + \dfrac{1}{\vartheta_1}(\mathbf{v}\cdot\nabla)\mathbf{v}_2 - \dfrac{\vartheta}{\vartheta_1\,\vartheta_2}(\mathbf{v}_2\cdot\nabla)\mathbf{v}_2 \\
-\mathrm{div}\,(\mu(\vartheta_1)\nabla\mathbf{v}) - \mathrm{div}\,((\mu(\vartheta_1)-\mu(\vartheta_2)\nabla\mathbf{v}_2)) + \dfrac{\lambda}{\vartheta_1}(\nabla\mathbf{v}_1 - \nabla^t\mathbf{v}_1)\nabla\vartheta \\
+\dfrac{\lambda}{\vartheta_1}(\nabla\mathbf{v}-\nabla^t\mathbf{v})\nabla\vartheta_2 - \dfrac{\lambda\,\vartheta}{\vartheta_1\,\vartheta_2}(\nabla\mathbf{v}_2-\nabla^t\mathbf{v}_2)\nabla\vartheta_2 + \nabla p = -\dfrac{\vartheta}{\vartheta_1\,\vartheta_2}\mathbf{f}, \\
\mathrm{div}\,\mathbf{v} = 0, \qquad \mathbf{v}_0(0,\cdot)=0, \quad \mathbf{v}|_{\partial\Omega}=0.
\end{cases}
\tag{57}
$$

We multiply (56) by $\lambda^2(-\Delta\vartheta+\vartheta)$ and (57) by $M\,\mathbf{v}$. Thanks to standard inequalities, we obtain:

$$
a'(t) + b(t) \le a(t)\,c(t),
$$

with

$$
\begin{aligned}
a(t) &= M\left\|\frac{\mathbf{v}}{\sqrt{\vartheta_1}}\right\|_{L^2}^2 + \lambda^2\,\|\vartheta\|_{H^1}^2\,,\\
b(t) &= \lambda^3\,m\,\|\Delta\vartheta\|_{L^2}^2 + \lambda\,M\,|\ln M|\,\|\nabla\mathbf{v}\|_{L^2}^2\,,\\
c(t) &= \frac{C\,M^3}{\lambda^3\,\min(1,|\ln M|^3)\,m^6}\Bigg(\frac{\lambda^2\,m^2}{M^2}\,\|\partial_t\vartheta_1\|_{H^1}^2 + \|\partial_t\mathbf{v}_2\|_{L^2}^2 + \|\nabla\mathbf{v}_1\|_{L^2}^4 + \|\nabla\mathbf{v}_2\|_{L^2}^4\\
&\qquad + \|\Delta\vartheta_1\|_{L^2}^4 + \|\Delta\vartheta_2\|_{L^2}^4 + \|\mathbf{f}\|_{L^2}^2\Bigg) + C\,\lambda\,(\|\Delta\vartheta_1\|_{L^2} + \|\Delta\vartheta_2\|_{L^2} + m)\,.
\end{aligned}
$$

As $a(0)=0$, the Gronwall's Lemma implies that $a(t)=0$ for any $t\in[0,T]$. We thus deduce that $\vartheta_1=\vartheta_2$ and $\mathbf{v}_1=\mathbf{v}_2$. With De Rham's theorem, uniqueness of p at zero average is straightforward.

4.2 Proof of Theorem 2

In order to get (23), (24) and (25), we only need to prove the following estimates:

$$
\left\|\vartheta^{(n,s)}(t)\right\|_{H^2}^2 + \int_0^t \left(\left\|\vartheta^{(n,s)}(s)\right\|_{H^3}^2 + \left\|\partial_t\vartheta^{(n,s)}(s)\right\|_{H^1}^2\right)\mathrm{d}s \le D\left[\frac{(B\,t)^n}{\sqrt{n!}}\right]^{\frac{1}{2}}
\tag{58}
$$

$$
\left\|\mathbf{v}^{(n,s)}(t)\right\|_{H^1}^2 + \int_0^t \left(\left\|\mathbf{v}^{(n,s)}(s)\right\|_{H^2}^2 + \left\|\partial_t\mathbf{v}^{(n,s)}(s)\right\|_{L^2}^2\right)\mathrm{d}s \le D\left[\frac{(B\,t)^n}{\sqrt{n!}}\right]^{\frac{1}{2}},
\tag{59}
$$

$$
\int_0^t \left\|p^{(n,s)}(s)\right\|_{L^2}^2\,\mathrm{d}s \le D\left[\frac{(B\,t)^n}{\sqrt{n!}}\right]^{\frac{1}{2}}.
\tag{60}
$$

Taking the gradient of (50) multiplied by $\eta = \frac{\delta}{m}\nabla\partial_t\vartheta^{(n,s)} - \lambda\nabla\Delta\vartheta^{(n,s)}$ with δ a positive parameter, we get:

$$
\begin{aligned}
&\frac{\delta}{m}\left\|\nabla\partial_t\vartheta^{(n,s)}\right\|_{L^2}^2 + \frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta\vartheta^{(n,s)}\right\|_{L^2}^2 + \lambda^2\int_\Omega \vartheta^{n-1}|\nabla\Delta\vartheta^{(n,s)}|^2\\
=&\,\frac{\delta\,\lambda}{m}\int_\Omega \vartheta^{n-1}\,\nabla\Delta\vartheta^{(n,s)}\cdot\nabla\partial_t\vartheta^{(n,s)} - \int_\Omega \nabla(\mathbf{v}^{(n-1,s)}\cdot\nabla\vartheta^{n+s})\cdot\eta - \int_\Omega \nabla(\mathbf{v}^{n-1}\cdot\nabla\vartheta^{(n,s)})\cdot\eta\\
&+\lambda\int_\Omega \nabla(\vartheta^{(n-1,s)}\Delta\vartheta^{n+s})\cdot\eta + \lambda\int_\Omega \Delta\vartheta^{(n,s)}\,\nabla\vartheta^{n-1}\cdot\eta - \lambda\int_\Omega \nabla(\nabla\vartheta^{(n-1,s)}\cdot\nabla\vartheta^{n+s})\cdot\eta\\
&-\lambda\int_\Omega \nabla(\nabla\vartheta^{n-1}\cdot\nabla\vartheta^{(n,s)})\cdot\eta.
\end{aligned}
$$

From (36) and usual inequalities, we obtain

$$
\begin{aligned}
&\frac{\delta}{2\,m}\left\|\nabla\partial_t\vartheta^{(n,s)}\right\|_{L^2}^2+\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta\vartheta^{(n,s)}\right\|_{L^2}^2+\lambda^2\,m\left(\frac{3}{4}-\frac{7\,\delta\,M^2}{2\,m^2}\right)\left\|\nabla\Delta\vartheta^{(n,s)}\right\|_{L^2}^2\\
\le&\frac{C_\delta}{m}\left\{\left\|\nabla(\mathbf{v}^{(n-1,s)}\cdot\nabla\vartheta^{n+s})\right\|_{L^2}^2+\left\|\nabla(\mathbf{v}^{n-1}\cdot\nabla\vartheta^{(n,s)})\right\|_{L^2}^2\right.\\
&\qquad+\lambda^2\left\|\nabla(\vartheta^{(n-1,s)}\Delta\vartheta^{n+s})\right\|_{L^2}^2+\lambda^2\left\|\Delta\vartheta^{(n,s)}\,\nabla\vartheta^{n-1}\right\|_{L^2}^2\\
&\qquad\left.+\lambda^2\left\|\nabla(\nabla\vartheta^{(n-1,s)}\cdot\nabla\vartheta^{n+s})\right\|_{L^2}^2+\lambda^2\left\|\nabla(\nabla\vartheta^{n-1}\cdot\nabla\vartheta^{(n,s)})\right\|_{L^2}^2\right\}.\\
\le&\frac{C_\delta}{m}\left\{\left\|\Delta\vartheta^{n+s}\right\|_{L^2}\left\|\nabla\Delta\vartheta^{n+s}\right\|_{L^2}\left\|\nabla\mathbf{v}^{(n-1,s)}\right\|_{L^2}^2+\left\|\mathbb{A}\mathbf{v}^{n-1}\right\|_{L^2}^2\left\|\Delta\vartheta^{(n,s)}\right\|_{L^2}^2\right.\\
&\qquad+\lambda^2\left\|\Delta\vartheta^{n+s}\right\|_{L^2}\left\|\nabla\Delta\vartheta^{n+s}\right\|_{L^2}\left\|\Delta\vartheta^{(n-1,s)}\right\|_{L^2}^2\\
&\qquad+\lambda^2\left\|\nabla\Delta\vartheta^{n+s}\right\|_{L^2}^2\left\|\vartheta^{(n-1,s)}\right\|_{H^1}\left\|\vartheta^{(n-1,s)}\right\|_{H^2}+\lambda^2\left\|\nabla\Delta\vartheta^{n-1}\right\|_{L^2}^2\left\|\Delta\vartheta^{(n,s)}\right\|_{L^2}^2\\
&\qquad\left.+\lambda^2\left\|\Delta\vartheta^{n+s}\right\|_{L^2}\left\|\nabla\Delta\vartheta^{n+s}\right\|_{L^2}\left\|\Delta\vartheta^{(n-1,s)}\right\|_{L^2}^2+\lambda^2\left\|\nabla\Delta\vartheta^{n-1}\right\|_{L^2}^2\left\|\Delta\vartheta^{(n,s)}\right\|_{L^2}^2\right\}.
\end{aligned}
$$

With $\delta=\frac{m^2}{14\,M^2}$, we obtain:

$$
\begin{aligned}
&\frac{m}{28\,M^2}\left\|\nabla\partial_t\vartheta^{(n,s)}\right\|_{L^2}^2+\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta\vartheta^{(n,s)}\right\|_{L^2}^2+\frac{\lambda^2\,m}{2}\left\|\nabla\Delta\vartheta^{(n,s)}\right\|_{L^2}^2\\
\le&\frac{C}{m}\left\{\left(\left\|\mathbb{A}\mathbf{v}^{n-1}\right\|_{L^2}^2+\lambda^2\left\|\nabla\Delta\vartheta^{n-1}\right\|_{L^2}^2\right)\left\|\Delta\vartheta^{(n,s)}\right\|_{L^2}^2\right.\\
&\qquad+\left\|\Delta\vartheta^{n+s}\right\|_{L^2}\left\|\nabla\Delta\vartheta^{n+s}\right\|_{L^2}\left(\lambda^2\left\|\Delta\vartheta^{(n-1,s)}\right\|_{L^2}^2+\left\|\nabla\mathbf{v}^{(n-1,s)}\right\|_{L^2}^2\right)\\
&\qquad\left.+\lambda^2\left\|\nabla\Delta\vartheta^{n+s}\right\|_{L^2}^2\left\|\vartheta^{(n-1,s)}\right\|_{H^1}\left\|\vartheta^{(n-1,s)}\right\|_{H^2}\right\}.
\end{aligned}
\tag{61}
$$

We now multiply (51) by $\mathbf{w}=\partial_t\mathbf{v}^{(n,s)}+\delta\mathbb{A}\mathbf{v}^{(n,s)}$ with δ a positive parameter. We obtain:

$$
\begin{aligned}
&\int_\Omega\frac{|\partial_t\mathbf{v}^{(n,s)}|^2}{\vartheta^n}+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\mu(\vartheta^n)|\nabla\mathbf{v}^{(n,s)}|^2+\delta\int_\Omega\mu(\vartheta^n)|\mathbb{A}\mathbf{v}^{(n,s)}|^2\\
=&-\delta\int_\Omega\frac{\partial_t\mathbf{v}^{(n,s)}}{\vartheta^n}\cdot\mathbb{A}\mathbf{v}^{(n,s)}+\frac{1}{2}\int_\Omega\mu'(\vartheta^n)\,\partial_t\vartheta^n\,|\nabla\mathbf{v}^{(n,s)}|^2+\delta\int_\Omega\mu'(\vartheta^n)\,q^{(n,s)}\,\nabla\vartheta^n\cdot\mathbb{A}\mathbf{v}^{(n,s)}\\
&+\delta\int_\Omega\mu'(\vartheta^n)\,(\nabla\vartheta^n\cdot\nabla)\mathbf{v}^{(n,s)}\cdot\mathbb{A}\mathbf{v}^{(n,s)}+\int_\Omega\frac{\vartheta^{(n,s)}}{\vartheta^{n+s}\,\vartheta^n}\partial_t\mathbf{v}^{n+s}\cdot\mathbf{w}\\
&+\int_\Omega\operatorname{div}\left((\mu(\vartheta^{n+s})-\mu(\vartheta^n))\nabla\mathbf{v}^{n+s}\right)\cdot\mathbf{w}+\int_\Omega\frac{\vartheta^{(n,s)}}{\vartheta^{n+s}\,\vartheta^n}(\mathbf{v}^{n+s-1}\cdot\nabla)\mathbf{v}^{n+s}\cdot\mathbf{w}\\
&-\int_\Omega\frac{1}{\vartheta^n}(\mathbf{v}^{(n-1,s)}\cdot\nabla)\mathbf{v}^{n+s}\cdot\mathbf{w}-\int_\Omega\frac{1}{\vartheta^n}(\mathbf{v}^{n-1}\cdot\nabla)\mathbf{v}^{(n,s)}\cdot\mathbf{w}\\
&+\lambda\int_\Omega\frac{\vartheta^{(n,s)}}{\vartheta^{n+s}\,\vartheta^n}(\nabla\mathbf{v}^{n+s}-\nabla^t\mathbf{v}^{n+s})\,\nabla\vartheta^{n+s}\cdot\mathbf{w}-\lambda\int_\Omega\frac{1}{\vartheta^n}(\nabla\mathbf{v}^{(n,s)}-\nabla^t\mathbf{v}^{(n,s)})\,\nabla\vartheta^{n+s}\cdot\mathbf{w}\\
&-\lambda\int_\Omega\frac{1}{\vartheta^n}(\nabla\mathbf{v}^n-\nabla^t\mathbf{v}^n)\,\nabla\vartheta^{(n,s)}\cdot\mathbf{w}-\int_\Omega\frac{\vartheta^{(n,s)}}{\vartheta^{n+s}\,\vartheta^n}\mathbf{f}\cdot\mathbf{w},
\end{aligned}
$$

where for $n\ge 1$, $q^n\in L^\infty(H^1(\Omega)\cap L^2_0(\Omega))$ is such that $-\Delta\mathbf{v}^n=\mathbb{A}\mathbf{v}^n+\nabla q^n$. Let ε be a small positive parameter. Similarly to (47), we get:

$$\left|\delta\int_\Omega \mu'(\vartheta^n)\,q^{(n,s)}\,\nabla\vartheta^n\cdot\mathbb{A}\mathbf{v}^{(n,s)}\right|$$
$$\leq 2\,\varepsilon\,\delta\,\lambda\,|\ln M|\,\left\|\mathbb{A}\mathbf{v}^{(n,s)}\right\|_{L^2}^2 + C_\varepsilon\frac{\delta\,\lambda\,\|\Delta\vartheta^n\|_{L^2}^4}{m^4\,|\ln M|^3}\left(1+\frac{\|\Delta\vartheta^n\|_{L^2}^4}{|\ln M|^4\,m^4}\right)\left\|\nabla\mathbf{v}^{(n,s)}\right\|_{L^2}^2.$$

Furthermore, one has:

$$\begin{aligned}\left|\int_\Omega \operatorname{div}\left((\mu(\vartheta^{n+s})-\mu(\vartheta^n))\nabla\mathbf{v}^{n+s}\right)\cdot\mathbf{w}\right| \leq & \frac{\left\|\partial_t\mathbf{v}^{(n,s)}\right\|_{L^2}^2}{20\,M} + \varepsilon\,\delta\,\lambda\,|\ln M|\,\left\|\mathbb{A}\mathbf{v}^{(n,s)}\right\|_{L^2}^2 \\ & + C_\varepsilon\left(M+\frac{\delta}{\lambda\,|\ln M|}\right)\left\|\operatorname{div}\left((\mu(\vartheta^{n+s})-\mu(\vartheta^n))\nabla\mathbf{v}^{n+s}\right)\right\|_{L^2}^2,\end{aligned}$$

and the mean value theorem and (36) allow us to obtain:

$$\begin{aligned}\left\|\operatorname{div}\left((\mu(\vartheta^{n+s})-\mu(\vartheta^n))\nabla\mathbf{v}^{n+s}\right)\right\|_{L^2}^2 &\leq \left\|(\mu(\vartheta^{n+s})-\mu(\vartheta^n))\nabla\mathbf{v}^{n+s}\right\|_{H^1}^2 \\ &\leq \frac{\lambda^2}{m^2}\left\|\vartheta^{(n,s)}\nabla\mathbf{v}^{n+s}\right\|_{H^1}^2 \leq \frac{C\,\lambda^2}{m^2}\,\|\mathbb{A}\mathbf{v}^{n+s}\|_{L^2}^2\left\|\vartheta^{(n,s)}\right\|_{H^2}^2.\end{aligned}$$

With ε small enough, we thus obtain:

$$\begin{aligned}&\frac{\left\|\partial_t\mathbf{v}^{(n,s)}\right\|_{L^2}^2}{2\,M}+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\mu(\vartheta^n)|\nabla\mathbf{v}^{(n,s)}|^2+\left(\frac{3}{4}-\frac{5\,\delta\,M}{\lambda\,|\ln M|\,m^2}\right)\delta\,\lambda\,|\ln M|\,\left\|\mathbb{A}\mathbf{v}^{(n,s)}\right\|_{L^2}^2 \\ \leq & C\Bigg\{\frac{\lambda}{\delta\,m^2\,|\ln M|}\,\|\partial_t\vartheta^n\|_{H^1}^2\left\|\nabla\mathbf{v}^{(n,s)}\right\|_{L^2}^2+\frac{\delta\,\lambda\,\|\Delta\vartheta^n\|_{L^2}^4}{m^4\,|\ln M|^3}\left(1+\frac{\|\Delta\vartheta^n\|_{L^2}^4}{|\ln M|^4\,m^4}\right)\left\|\nabla\mathbf{v}^{(n,s)}\right\|_{L^2}^2 \\ &\quad+\frac{\delta\,\lambda}{m^4\,|\ln M|^3}\,\|\Delta\vartheta^n\|_{L^2}^4\left\|\nabla\mathbf{v}^{(n,s)}\right\|_{L^2}^2\Bigg\} \\ &+C\left(M+\frac{\delta}{\lambda\,|\ln M|}\right)\Bigg\{\frac{1}{m^4}\,\|\partial_t\mathbf{v}^{n+s}\|_{L^2}^2\left\|\vartheta^{(n,s)}\right\|_{H^2}^2+\frac{\lambda^2}{m^2}\,\|\mathbb{A}\mathbf{v}^{n+s}\|_{L^2}^2\left\|\vartheta^{(n,s)}\right\|_{H^2}^2 \\ &\qquad+\frac{1}{m^4}\,\|\nabla\mathbf{v}^{n+s-1}\|_{L^2}^2\,\|\mathbb{A}\mathbf{v}^{n+s}\|_{L^2}^2\left\|\vartheta^{(n,s)}\right\|_{H^1}^2 \\ &\qquad+\frac{1}{m^2}\,\|\nabla\mathbf{v}^{n+s}\|_{L^2}\,\|\mathbb{A}\mathbf{v}^{n+s}\|_{L^2}\left\|\nabla\mathbf{v}^{(n-1,s)}\right\|_{L^2}^2 \\ &\qquad+\frac{M+\frac{\delta}{\lambda\,|\ln M|}}{m^4\,\delta\,\lambda\,|\ln M|}\,\|\nabla\mathbf{v}^{n-1}\|_{L^2}^4\left\|\nabla\mathbf{v}^{(n,s)}\right\|_{L^2}^2 \\ &\qquad+\frac{\lambda^2}{m^4}\,\|\Delta\vartheta^{n+s}\|_{L^2}^2\,\|\mathbb{A}\mathbf{v}^{n+s}\|_{L^2}^2\left\|\vartheta^{(n,s)}\right\|_{H^1}^2 \\ &\qquad+\frac{\left(M+\frac{\delta}{\lambda\,|\ln M|}\right)\lambda^3}{m^4\,\delta\,|\ln M|}\,\|\Delta\vartheta^{n+s}\|_{L^2}^4\left\|\nabla\mathbf{v}^{(n,s)}\right\|_{L^2}^2 \\ &\qquad+\frac{\lambda^2}{m^2}\,\|\mathbb{A}\mathbf{v}^n\|_{L^2}^2\left\|\Delta\vartheta^{(n,s)}\right\|_{L^2}^2+\frac{1}{m^4}\,\|\mathbf{f}\|_{L^2}^2\left\|\vartheta^{(n,s)}\right\|_{H^2}^2.\Bigg\}\end{aligned}$$

We choose $\delta=\frac{\lambda\,|\ln M|\,m^2}{20\,M}$. To simplify the expression, we note that thanks to $\dfrac{m^2}{M}\leq M$, then $M+$

$\dfrac{\delta}{\lambda\,|\ln M|} \le C\,M$. We also use that $\|\vartheta\|_{H^2} \sim \|\vartheta\|_{H^1} + \|\Delta\vartheta\|_{L^2}$ on $H^2_N(\Omega)$ in order to get:

$$
\begin{aligned}
&\frac{\left\|\partial_t \mathbf{v}^{(n,s)}\right\|_{L^2}^2}{2\,M} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mu(\vartheta^n)|\nabla\mathbf{v}^{(n,s)}|^2 + \frac{\lambda^2\,|\ln M|^2\,m^2}{40\,M}\left\|\mathbb{A}\mathbf{v}^{(n,s)}\right\|_{L^2}^2 \\
\le &C\Bigg\{\frac{M}{m^4|\ln M|^2}\,\|\partial_t\vartheta^n\|_{H^1}^2 + \frac{\lambda^2\,\|\Delta\vartheta^n\|_{L^2}^4}{m^2\,M\,|\ln M|^2}\left(1+\frac{\|\Delta\vartheta^n\|_{L^2}^4}{|\ln M|^4\,m^4}\right) + \frac{M}{\lambda^2\,m^4}\,\|\partial_t\mathbf{v}^{n+s}\|_{L^2}^2 \\
&\qquad + \frac{M}{m^2}\,\|\mathbb{A}\mathbf{v}^{n+s}\|_{L^2}^2 + \frac{M^3}{m^6\,\lambda^2\,|\ln M|^2}\,\|\nabla\mathbf{v}^{n-1}\|_{L^2}^4 \\
&\qquad + \frac{M^3\,\lambda^2}{m^6|\ln M|^2}\,\|\Delta\vartheta^{n+s}\|_{L^2}^4 + \frac{M}{m^2}\,\|\mathbb{A}\mathbf{v}^n\|_{L^2}^2 + \frac{M}{\lambda^2\,m^4}\,\|\mathbf{f}\|_{L^2}^2\Bigg\} \\
&\qquad\quad \times\left(\left\|\nabla\mathbf{v}^{(n,s)}\right\|_{L^2}^2 + \lambda^2\left\|\Delta\vartheta^{(n,s)}\right\|_{L^2}^2\right) \\
&+ C\frac{M}{m^2}\,\|\nabla\mathbf{v}^{n+s}\|_{L^2}\,\|\mathbb{A}\mathbf{v}^{n+s}\|_{L^2}\left\|\nabla\mathbf{v}^{(n-1,s)}\right\|_{L^2}^2 \\
&+ C\Bigg\{\frac{M}{m^4}\,\|\partial_t\mathbf{v}^{n+s}\|_{L^2}^2 + \frac{\lambda^2\,M}{m^2}\,\|\mathbb{A}\mathbf{v}^{n+s}\|_{L^2}^2 + \frac{M}{m^4}\,\|\nabla\mathbf{v}^{n+s-1}\|_{L^2}^2\,\|\mathbb{A}\mathbf{v}^{n+s}\|_{L^2}^2 \\
&\qquad + \frac{\lambda^2\,M}{m^4}\,\|\Delta\vartheta^{n+s}\|_{L^2}^2\,\|\mathbb{A}\mathbf{v}^{n+s}\|_{L^2}^2 + \frac{M}{m^4}\,\|\mathbf{f}\|_{L^2}^2\Bigg\}\left\|\vartheta^{(n,s)}\right\|_{H^1}^2 .
\end{aligned}
\tag{62}
$$

We multiply (61) by $2\,\lambda$ and (62) by $\dfrac{2}{\lambda\,|\ln M|}$ and we sum. By using

$$
\left\|\nabla\mathbf{v}^{(n,s)}\right\|_{L^2}^2 \le \frac{1}{\lambda\,|\ln M|}\left\|\sqrt{\mu(\vartheta^n)}\nabla\mathbf{v}^{(n,s)}\right\|_{L^2}^2 ,
$$

we obtain

$$
a_n'(t) + b_n(t) \le c_n(t)\,a_n(t) + d_n(t)\,a_{n-1}(t) + e_n(t),
$$

with:

$$
\begin{aligned}
a_n =&\lambda^2\left\|\Delta\vartheta^{(n,s)}\right\|_{L^2}^2 + \frac{1}{\lambda\,|\ln M|}\left\|\sqrt{\mu(\vartheta^n)}\nabla\mathbf{v}^{(n,s)}\right\|_{L^2}^2 \\
b_n =&\frac{\lambda\,m}{14\,M^2}\left\|\nabla\partial_t\vartheta^{(n,s)}\right\|_{L^2}^2 + \lambda^3\,m\left\|\nabla\Delta\vartheta^{(n,s)}\right\|_{L^2}^2 + \frac{\left\|\partial_t\mathbf{v}^{(n,s)}\right\|_{L^2}^2}{\lambda\,M\,|\ln M|} + \frac{\lambda\,|\ln M|\,m^2}{20\,M}\left\|\mathbb{A}\mathbf{v}^{(n,s)}\right\|_{L^2}^2 \\
c_n =&C\Bigg\{\frac{1}{\lambda\,m}\,\|\mathbb{A}\mathbf{v}^{n-1}\|_{L^2}^2 + \frac{\lambda}{m}\,\|\nabla\Delta\vartheta^{n-1}\|_{L^2}^2 + \frac{M}{\lambda\,m^4|\ln M|^3}\,\|\partial_t\vartheta^n\|_{H^1}^2 \\
&\qquad + \frac{\lambda\,\|\Delta\vartheta^n\|_{L^2}^4}{m^2\,M\,|\ln M|^3}\left(1+\frac{\|\Delta\vartheta^n\|_{L^2}^4}{|\ln M|^4\,m^4}\right) + \frac{M}{\lambda^3\,|\ln M|\,m^4}\,\|\partial_t\mathbf{v}^{n+s}\|_{L^2}^2 \\
&\qquad + \frac{M}{\lambda\,|\ln M|\,m^2}\,\|\mathbb{A}\mathbf{v}^{n+s}\|_{L^2}^2 + \frac{M^3}{m^6\,\lambda^3\,|\ln M|^3}\,\|\nabla\mathbf{v}^{n-1}\|_{L^2}^4 \\
&\qquad + \frac{M^3\,\lambda}{m^6|\ln M|^3}\,\|\Delta\vartheta^{n+s}\|_{L^2}^4 + \frac{M}{\lambda\,|\ln M|\,m^2}\,\|\mathbb{A}\mathbf{v}^n\|_{L^2}^2 + \frac{M}{\lambda^3\,|\ln M|\,m^4}\,\|\mathbf{f}\|_{L^2}^2\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
d_n =& C\left\{ \frac{\lambda}{m} \left\|\Delta \vartheta^{n+s}\right\|_{L^2} \left\|\nabla\Delta \vartheta^{n+s}\right\|_{L^2} + \frac{M}{\lambda\,|\ln M|\, m^2} \left\|\nabla \mathbf{v}^{n+s}\right\|_{L^2} \left\|\Delta \mathbf{v}^{n+s}\right\|_{L^2} \right\} \\
e_n =& C\frac{\lambda^3}{m} \left\|\nabla\Delta \vartheta^{n+s}\right\|_{L^2}^2 \left\|\vartheta^{(n-1,s)}\right\|_{H^2} \left\|\vartheta^{(n-1,s)}\right\|_{H^1} \\
&+ C\left\{ \frac{M}{m^4\,\lambda\,|\ln M|} \left\|\partial_t \mathbf{v}^{n+s}\right\|_{L^2}^2 + \frac{\lambda\, M}{m^2\,|\ln M|} \left\|\mathbb{A} \mathbf{v}^{n+s}\right\|_{L^2}^2 \right. \\
&\quad + \frac{M}{m^4\,\lambda\,|\ln M|} \left\|\nabla \mathbf{v}^{n+s-1}\right\|_{L^2}^2 \left\|\mathbb{A} \mathbf{v}^{n+s}\right\|_{L^2}^2 \\
&\quad \left. + \frac{\lambda\, M}{m^4\,|\ln M|} \left\|\Delta \vartheta^{n+s}\right\|_{L^2}^2 + \frac{M}{\lambda\,|\ln M|\, m^4} \left\|\mathbf{f}\right\|_{L^2}^2 \right\} \left\|\vartheta^{(n,s)}\right\|_{H^1}^2 .
\end{aligned}
$$

Thanks to Theorem 3, (c_n) is bounded in $L^1(0,T)$, (d_n) is bounded in $L^2(0,T)$ and $(e_n) \in L^1(0,T)$. Moreover, (e_n) satisfies

$$
\|e_n\|_{L^1(0,t)}^2 \leq D \left[\frac{(B\,t)^{n-1}}{(n-1)!} \right]^{\frac{1}{2}}, \tag{63}
$$

because of (52). We now apply Gronwall's Lemma and note that $a_n(0) = 0$ to get for $t \in (0,T)$:

$$
\begin{aligned}
a_n(t) + \int_0^t b_n(s)\,\mathrm{d}s &\leq D \left(\left[\frac{(B\,t)^{n-1}}{\sqrt{(n-1)!}} \right]^{\frac{1}{2}} + \|a_0\|_{L^\infty(0,t)} \left(\frac{(B\,t)^n}{n!} \right)^{\frac{1}{2}} \right) \\
&\leq D \left[\frac{(B\,t)^{n-1}}{\sqrt{(n-1)!}} \right]^{\frac{1}{2}},
\end{aligned}
$$

where $B > 0$ et $D > 0$ are constants not depending on n. This shows estimates (58) and (59). We obtain (60) by multiplying (51) by $\nabla p^{(n,s)}$.

Remark 2. *Note that the loss of convergence between estimates of Theorem 1 and Theorem 2 is due to the estimation of the term* $\vartheta^{(n-1,s)} \nabla\Delta \vartheta^{n+s}$*.This is a consequence of the non-linearity of the temperature equation.*

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